PROBLEM SET 1

SOLUTIONS

Astronomy 208 — Fall 2000

1. Estimate the magnitude distribution of stars in a 1° patch of sky

   To get the distribution of stars in the Galaxy, we can start with the IMF (Initial Mass Function, not International Monetary Fund). Note that the IMF is defined as the total number of stars *ever formed* per square parsec in the galaxy, which is different from the PDMF (Present Day Mass Function), which is the number of stars currently in existence per square parsec. The difference lies in the fact that some stars will have died since the formation of the galaxy, so using the IMF will overestimate the number of stars. For a constant birthrate of stars, the PDMF (ϕ) is related to the IMF (ξ) by

$$\phi(\log M) = \begin{cases} \xi(\log M) T_{ms}/T_0, & T_{ms} < T_0 \\ \xi(\log M), & T_{ms} \geq T_0 \end{cases} \tag{1}$$

   where the PMDF and IMF are both in terms of stars per square parsec per logarithmic mass interval, T_{ms} is the main sequence lifetime of a star, and T_0 is the age of the galaxy.

   We can model the IMF as a broken power law as in equation (32) of Miller & Scalo (1979) [henceforth MS],

$$\xi(\log M) \propto \begin{cases} M^{-0.4}, & 0.1 \leq M \leq 1 \\ M^{-1.5}, & 1 \leq M \leq 10 \\ M^{-2.3}, & 10 \leq M \leq 62 \end{cases} \tag{2}$$

   where M is in units of $M_\odot$ and $\xi(\log M)$ is the number of stars per square parcecs per log mass interval. I have used the values given in Table 7 of MS, excluding the normalization constants, since I will renormalize this function later.

   I will assume that $T_{ms} = T_0$ for 1 $M_\odot$, and that $T_{ms} \propto M^{-2.5}$ to get

$$T_{ms}/T_0 = M^{-2.5}. \tag{3}$$

   Using equation 2, we find that the PDMF is

$$\phi(\log M) \propto \begin{cases} M^{-0.4}, & 0.1 \leq M \leq 1 \\ M^{-4}, & 1 \leq M \leq 10 \\ M^{-4.8}, & 10 \leq M \leq 62. \end{cases} \tag{4}$$

   Now we need to convert this into a magnitude distribution. I will assume that $L/L_\odot = (M/M_\odot)^{3.5}$. Luminosity is related to absolute magnitude by $M_v = 4.72 - 2.5\log(L/L_\odot)$, so

$$M_v = 4.72 - 8.75\log(M/M_\odot). \tag{5}$$

   or

$$M/M_\odot = 10^{[(4.72-M_v)/8.75]}. \tag{6}$$

The conversion from $\phi(\log M)$ to $\phi(M_v)$ is

$$\phi(M_v) = \phi(\log M)\left|\frac{d\log M}{dM_v}\right| \tag{7}$$

Substituting in equation (4):

$$\phi(M_v) \propto \begin{cases} 10^{0.4M_v/8.75}, & 13.47 \geq M_v \geq 4.72 \\ 10^{4M_v/8.75}, & 4.72 \geq M_v \geq -4.03 \\ 10^{4.8M_v/8.75}, & -4.03 \geq M_v \geq -10.96. \end{cases} \tag{8}$$

(Note: a shortcut is to take the luminosity function of stars per unit absolute magnitude per pc^3 directly from Figure 1 of MS.)

If we normalize equation(8) so that $\int \phi(M_v)\,dM_v = 1$ and that the power-law segments are continuous at the boundaries, we find

$$\phi(M_v) = \begin{cases} 0.040 \times 10^{0.046M_v}, & 13.47 \geq M_v \geq 4.72 \\ (4.5 \times 10^{-4}) \times 10^{0.46M_v}, & 4.72 \geq M_v \geq -4.03 \\ (1.06 \times 10^{-3}) \times 10^{0.55M_v}, & -4.03 \geq M_v \geq -10.96 \end{cases} \tag{9}$$

In order to convert this from a distribution in absolute magnitude (M_v) to a distribution in apparent magnitude (m_v) and distance (d) we need the relation

$$M_v = m_v - 5\log(d/10\,\mathrm{pc}).$$

Since we eventually will be integrating this over the distance, I will rewrite the limits in terms of d:

$$\phi(m_v, d) = \begin{cases} 0.040 \times 10^{0.046m_v}(\frac{d}{10\,\mathrm{pc}})^{-0.23}, & 0.02 \times 10^{m_v/5} \leq d\,(\mathrm{pc}) \leq 1.14 \times 10^{m_v/5} \\ (4.5 \times 10^{-4})\,10^{0.46m_v}(\frac{d}{10\,\mathrm{pc}})^{-2.3}, & 1.14 \times 10^{m_v/5} \leq d\,(\mathrm{pc}) \leq 64 \times 10^{m_v/5} \\ (1.06 \times 10^{-3})\,10^{0.55m_v}(\frac{d}{10\,\mathrm{pc}})^{-2.7}, & 64 \times 10^{m_v/5} \leq d\,(\mathrm{pc}) \leq 1600 \times 10^{m_v/5} \end{cases} \tag{10}$$

Note that the ranges on d are dependent on m_v. The limits correspond to the minimum and maximum distances at which a star in a given mass range will have an apparent magnitude of m_v.

To get the total magnitude distribution, we need to integrate this over the volume of space seen in 1 square degree. This integral has the following form:

$$\phi_{\mathrm{tot}}(m_v) = \int_V \phi(m_v, d)\, n_\star(d)\, dV \tag{11}$$

where $n_\star$ is the number density of stars and $\phi(m_v)\,dm_v$ is the number of stars with apparent magnitude between m_v and $m_v + dm_v$.

To determine $n_\star$, I will assume that the stellar density decreases exponentially as a function of Galactic radius, and also falls off exponentially with distance from the midplane of the disk, so

$$n_\star = n_0 e^{-r/r_0} e^{-z/z_0} \tag{12}$$

where r_0 is the scale radius and z_0 is the half the scale height of the Galaxy. Using values from p. 680 of Binney & Merrifield (1998), $r_0 = 3$ kpc and $z_0 = 180$ pc. We will also assume that the sun is 8 kpc from the Galactic center. In the vicinity of the sun ($r = 8$ kpc and $z = 0$ pc), $n_\star \sim 0.5\,\mathrm{pc}^{-3}$. (I get this from the fact that the nearest star is about 1 parsec away, so the number density of stars should be bit less than 1 pc^{-3}.) Solving, I find $n_0 = 7.2\,\mathrm{pc}^{-3}$.

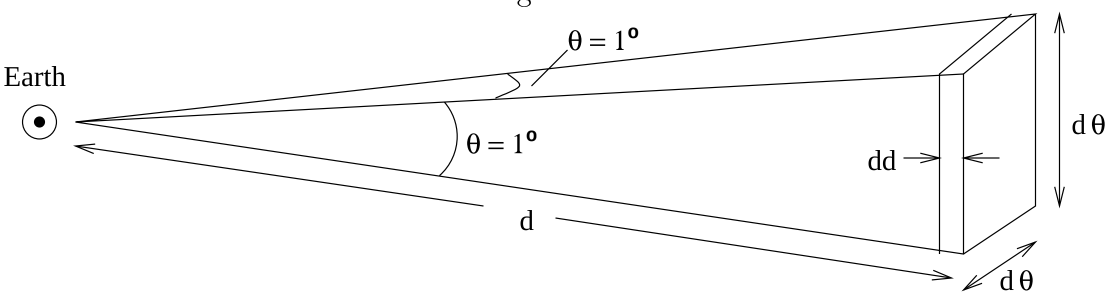
The volume integral is over the volume seen in one square degree. As we go further out in d, the cross-sectional area increases, as shown in Figure 1. The volume element dV can be rewritten as

$$dV = A\,dd = (d\theta)^2\,dd = \theta^2 d^2\,dd. \tag{13}$$

where $\theta = 1^\circ = \pi/180\,\mathrm{radians} = 0.0175$.

Some if you integrated $dV = d^2 \sin\theta\,d\theta\,d\phi\,dd$ with $0 \le \theta \le 0.0175$ and $0 \le \phi \le 0.0175$. *This is incorrect.* See the Appendix for the correct way to do it.

Figure 1:



Thus, equation (11) becomes

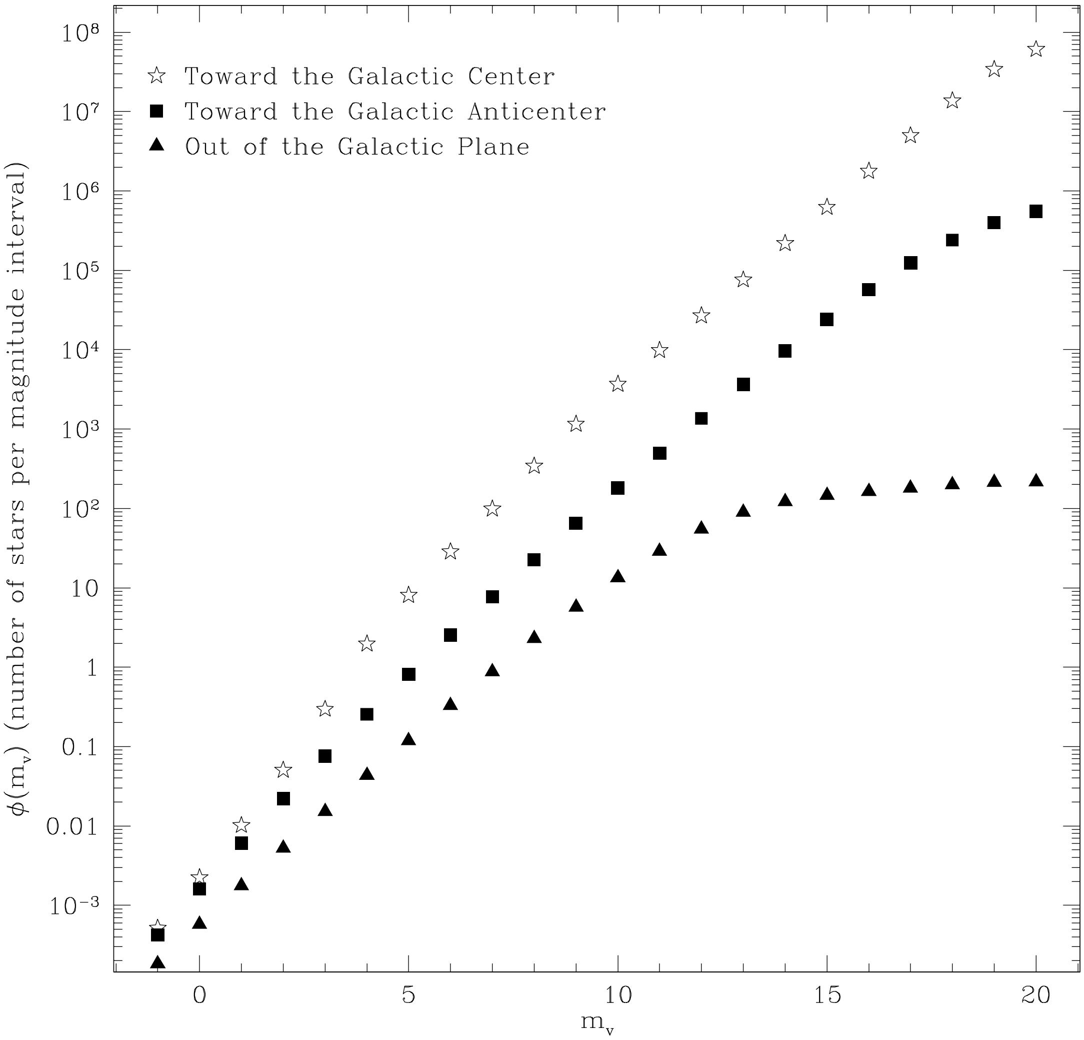
$$\begin{aligned}\phi_{\rm tot}(m_v) &= \int_{0.020\times 10^{m_v/5}}^{1.14\times 10^{m_v/5}} 0.040 \times 10^{0.046 m_v} \left(\frac{d}{10\,\mathrm{pc}}\right)^{-0.23} n_\star(d)\,\theta^2 d^2\,dd \\ &+ \int_{1.14\times 10^{m_v/5}}^{64\times 10^{m_v/5}} (4.5\times 10^{-4}) \times 10^{0.46 m_v} \left(\frac{d}{10\,\mathrm{pc}}\right)^{-2.3} n_\star(d)\,\theta^2 d^2\,dd \\ &+ \int_{64\times 10^{m_v/5}}^{1600\times 10^{m_v/5}} (1.06\times 10^{-3}) \times 10^{0.55 m_v} \left(\frac{d}{10\,\mathrm{pc}}\right)^{-2.7} n_\star(d)\,\theta^2 d^2\,dd \end{aligned} \tag{14}$$

where $n_\star$ depends on the line of sight.

(a) Toward the Galactic center:

If the distance from the earth/sun is d and the distance from the Galactic center is r, then as we move toward the Galactic center, $r = 8 - d$ kpc. Past the Galactic

Figure 2:



Toward the Galactic Center
Toward the Galactic Anticenter
Out of the Galactic Plane
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10
1
0.1
0.01
10^{-3}
$\phi(m_v)$ (number of stars per magnitude interval)
0
5
10
15
20
m_v

center, $r = d - 8$ kpc. Substituting into the equation for $n_\star$ and taking $z = 0$,

$$\begin{aligned} n_\star(d) &= \begin{cases} n_0\, e^{(d-8000)/3000}, & d < 8000\,\mathrm{pc} \\ n_0\, e^{(-d+8000)/3000}, & d \geq 8000\,\mathrm{pc} \end{cases} \\ &= 7.2\,\mathrm{pc}^{-3}\, e^{-|d-8000|/3000^4} \end{aligned} \tag{15}$$

Note: d needs to be converted to parsecs to be consistent with the units used in the integral.

The resulting distribution (via Mathematica) is plotted in Fig. 2 as stars.

(b) Toward the Galactic anticenter:

180° away from Galactic center, $d = r + 8$ kpc, so

$$n_\star(d) = 7.2\,\mathrm{pc}^{-3}\, e^{-(d+8000)/3000} \tag{16}$$

where d is in pc.

The resulting distribution (via Mathematica) is plotted in Fig. 2 as squares.

(c) Directly overhead:

Here, $r = 8$ kpc is constant and z changes:

$$n_\star = n_0\, e^{-0.8} e^{-z/z_0} = 0.5\mathrm{cm}^{-3}\, e^{-z/z_0} \tag{17}$$

where z_0 is 180 pc.

The resulting distribution (via Mathematica) is plotted in Fig. 2 as triangles.

2. Estimate the number of stars per square degree to a limiting magnitude of $m_v = 14\,\mathrm{mag}$

For this problem, we just integrate the functions calculated in Problem 1 for $m_v \leq 14,\mathrm{mag}$. I will also choose an arbitrary minimum magnitude of $m_v \geq -1$, since the brightest star visible from earth has a visual magnitude of -1. (Really, if your lower limit is between -1 and $-\infty$, your answer shouldn't be affected too much.)

(a) Toward the Galactic center:

$$\boxed{2.1 \times 10^5 \text{ stars per square degree.}}$$

(b) 180° away from Galactic center:

$$\boxed{9800 \text{ stars per square degree.}}$$

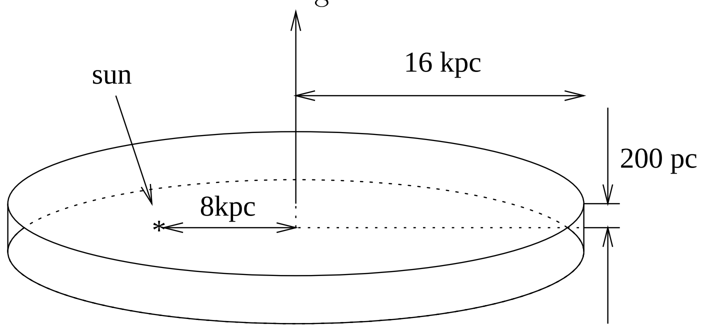
(c) Directly overhead:

$$\boxed{260 \text{ stars per square degree.}}$$

3. Estimate the HI column density

Let's assume that the average HI number density in the solar neighborhood is $1\,\mathrm{cm}^{-3}$. From §9.2 of Binney & Merrifield, the distribution of HI is approximately uniform out to 16 kpc from the center of the Galaxy. So, let's assume that the gas in uniform in

Figure 3:



the radial direction, and falls off exponentially with radius with a scale height of 200 pc. (See Fig. 3 for illustration.) Then,

$$n_H = (1\,\mathrm{cm}^{-3}) \times e^{-z/200\,\mathrm{pc}}. \tag{18}$$

The column density along a given line of sight is

$$N_H = \int_\ell n_H \, d\ell \tag{19}$$

where ℓ is the length of the column.

(a) Toward the Galactic center
In the Galactic plane, n_H is constant. Thus,

$$N_H = n_H \ell = 1\,\mathrm{cm}^{-3} \times \ell \times \frac{3.1 \times 10^{18}\mathrm{cm}}{\mathrm{pc}} = (3.1 \times 10^{18}\mathrm{cm}^{-2})\ \ell[\mathrm{pc}] \tag{20}$$

The total column length toward the Galactic center is $8\,\mathrm{kpc} + 16\,\mathrm{kpc} = 24\,\mathrm{kpc}$. So the total column density is

$$N_H = (3.1 \times 10^{18}\mathrm{cm}^{-2}) \times (24,000) = \boxed{7.4 \times 10^{22}\mathrm{cm}^{-2}.}$$

(b) 180° away from Galactic center
This line of sight is also in the Galactic plane so we can use Equation (20). Away from the Galactic center, $\ell = 8\,\mathrm{kpc}$ so

$$N_H = (3.1 \times 10^{18}\mathrm{cm}^{-2}) \times (8000) = \boxed{2.5 \times 10^{22}\mathrm{cm}^{-2}.}$$

(c) Directly overhead
Out of the plane of the Galaxy, $\ell = 200\,\mathrm{pc}$ so

$$N_H = \int_\ell n_H \, d\ell = \int_0^\infty (1\,\mathrm{cm}^{-3}) e^{-z/200\,\mathrm{pc}} \, d\ell = (1\,\mathrm{cm}^{-3})(200\,\mathrm{pc}) = \boxed{6.2 \times 10^{20}\mathrm{cm}^{-2}.}$$

4. Estimate the extinction, using

$$A_v[\text{mag}] = N_H \times 10^{-21}[\text{cm}^{-2}] \tag{21}$$

(a) Per kpc:

i. & ii. In the plane of the Galaxy, we derive the extinction per kpc from Equation (20). In 1 kpc, the column density is

$$N_H(1\,\text{kpc}) = (3.1 \times 10^{18}\text{cm}^{-2}) \times 1000 = 3.1 \times 10^{21}\ \text{cm}^{-2}$$

so the extinction per kpc is

$$\frac{A_v}{1\,\text{kpc}} = \frac{N_H}{1\,\text{kpc}} \times 10^{-21} = \boxed{3.1\ \text{mag/kpc}}$$

toward both the Galactic center and anticenter.

iii. Out of the plane of the Galaxy, we can define the extinction per kpc as $dA_v/d\ell$ where ℓ is in units of kpc.

$$\begin{aligned}
A_v &= N_H \times 10^{-21}(\text{mag/cm}^{-2}) = \int_0^\ell n_H\, d\ell' \times 10^{-21}(\text{mag/cm}^{-2}) \\
&= (1\,\text{cm}^{-3}) \times 10^{-21}(\text{mag/cm}^{-2}) \times \int_0^\ell e^{-z/200\,\text{pc}}\, dz \\
&= (1 \times 10^{-21}\text{mag}\,\text{cm}^{-1}) \times (200\,\text{pc})(1 - e^{-\ell/200\,\text{pc}}) \\
&= (0.62\,\text{mag})(1 - e^{-\ell/0.2\,\text{kpc}}) \\
\frac{dA_v}{d\ell} &= \frac{0.62\,\text{mag}}{0.2\,\text{kpc}} e^{-\ell/0.2\,\text{kpc}} \\
&= \boxed{3.1\, e^{-\ell/0.2\,\text{kpc}}\ \text{mag/kpc}}
\end{aligned}$$

(b) in total:

For this part, we can simply take the total column densities calculated in Problem 3 and plug into Equation (21).

i. Toward the Galactic center
$N_H = 7.4 \times 10^{22}\text{cm}^{-2}$ so

$$A_v = 7.4 \times 10^{22} \times 10^{-21} = \boxed{74\ \text{mag}}$$

ii. 180° away from Galactic center
$N_H = 2.5 \times 10^{22}\text{cm}^{-2}$ so

$$A_v = 2.5 \times 10^{22} \times 10^{-21} = \boxed{25\ \text{mag}}$$

iii. Directly overhead
$N_H = 6.2 \times 10^{20}\text{cm}^{-2}$ so

$$A_v = 6.2 \times 10^{20} \times 10^{-21} = \boxed{0.62\ \text{mag}}$$

5. Suppose Barnard photgraphed the Taurus dark clouds with a limiting magnitude of $m_v = 14\,\text{mag}$.

First, we should find the position of the Taurus clouds in terms of galactic coordinates. Taurus is at $\alpha = 4^h\,20^m, \delta = 25°$ and the conversion to galactic coordinates is

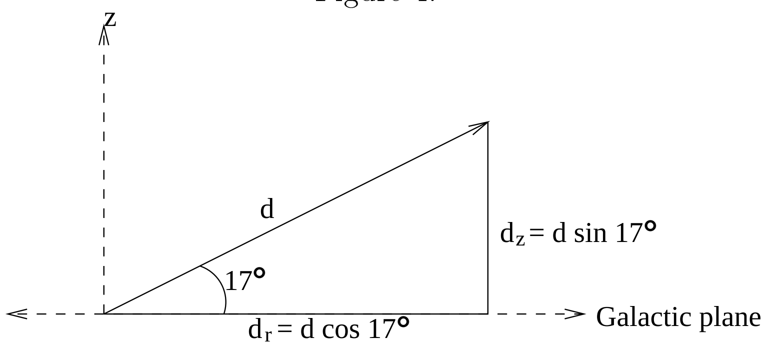
$$\sin b = \sin\delta_{GP}\sin\delta + \cos\delta_{GP}\cos\delta\cos(\alpha - \alpha_{GP}) \tag{22}$$
$$\cos b\sin(l_{CP} - l) = \cos\delta\sin(\alpha - \alpha_{GP}) \tag{23}$$
$$\cos b\cos(l_{CP} - l) = \cos\delta_{GP}\sin\delta - \sin\delta_{GP}\cos\delta\cos(\alpha - \alpha_{GP}) \tag{24}$$
$$\tag{25}$$

where (l, b) are Galactic longitude and latitude, respectively, $(\alpha_{GP}, \delta_{GP}) = (192.9°, 27.1°)$ is the RA and declination of the North Galactic Pole and $l_{CP} = 123.932°$ is the Galactic longitude of the North Celestial Pole. Solving, we find that the Taurus clouds are at $l = 173°, b = -17°$, indicating that Taurus is *away* from the Galactic center ($l = 0°$ points toward the Galactic center).

(a) How many stars per square degree could he expect without the "extra" extinction due to his "dark nebulae?"

This calculation is similar to Problem 2, but we have to take two things into account: (1) the line of sight, which is non-trivial, and (2) extinction as described in Problem 4.

Figure 4:



The line of sight is 17° out of the plane of the Galaxy, and 173° offset from the Galactic center (see Fig. 4). We can decompose d into its projections onto the plane of the galaxy and the z-axis. From Fig. 4, $d_z = d\sin 17°$ and $d_r = d\cos 17°$. In particular,

$$z = d_z = d\sin 17°.$$

To calculate r, you can do some more trigonometry, saying that r is the third side of a triangle with sides 8 kpc and d_r with an interior angle of 173°, and get

$$r = \sqrt{(8\,\text{kpc})^2 + d_r^2 - 2(8\,\text{kpc})d_r\cos 173°}$$

or you can say that 173° is awfully close to 180°, and

$$r = d_r + 8\,\mathrm{kpc} = d\cos 17^\circ + 8\,\mathrm{kpc}.$$

I will opt for the latter. Thus $n_\star$ is

$$n_\star(d) = n_0 e^{-r/r_0 - z/z_0} = (7.2\,\mathrm{pc}^{-3})\exp\left[-\frac{d\cos 17^\circ + 8\,\mathrm{kpc}}{3\,\mathrm{kpc}} - \frac{d\sin 17^\circ}{180\,\mathrm{pc}}\right]$$

The increase in apparent magnitude due to extinction is simply the A_v calculated in Problem 4. So,

$$m_v = M_v + 5\log(d/10\,\mathrm{pc}) + 3.1(d/1\,\mathrm{kpc}). \tag{26}$$

The integral we need to solve looks like

$$N_\star = \int_{-1}^{14} \phi_(m_v)\, dm_v = \int_{-1}^{14}\int_d \phi(m_v, d)\, n_\star(d)\, \theta^2 d^2\, dd\, dm_v. \tag{27}$$

After plugging into my numerical integrator (Mathematica) I find

$\boxed{\text{1460 stars per square degree}}$ with *no* extinction
and
$\boxed{\text{580 stars per square degree}}$ including extinction.

(b) How much extinction would be associated with "almost black" and "totally black" regions?

I have plotted the cumulative number of stars visible out to a a magnitude m_v in Figure 5. Additional extinction due to the Taurus clouds has the effect of shifting this curve to the right, or to higher magnitude.

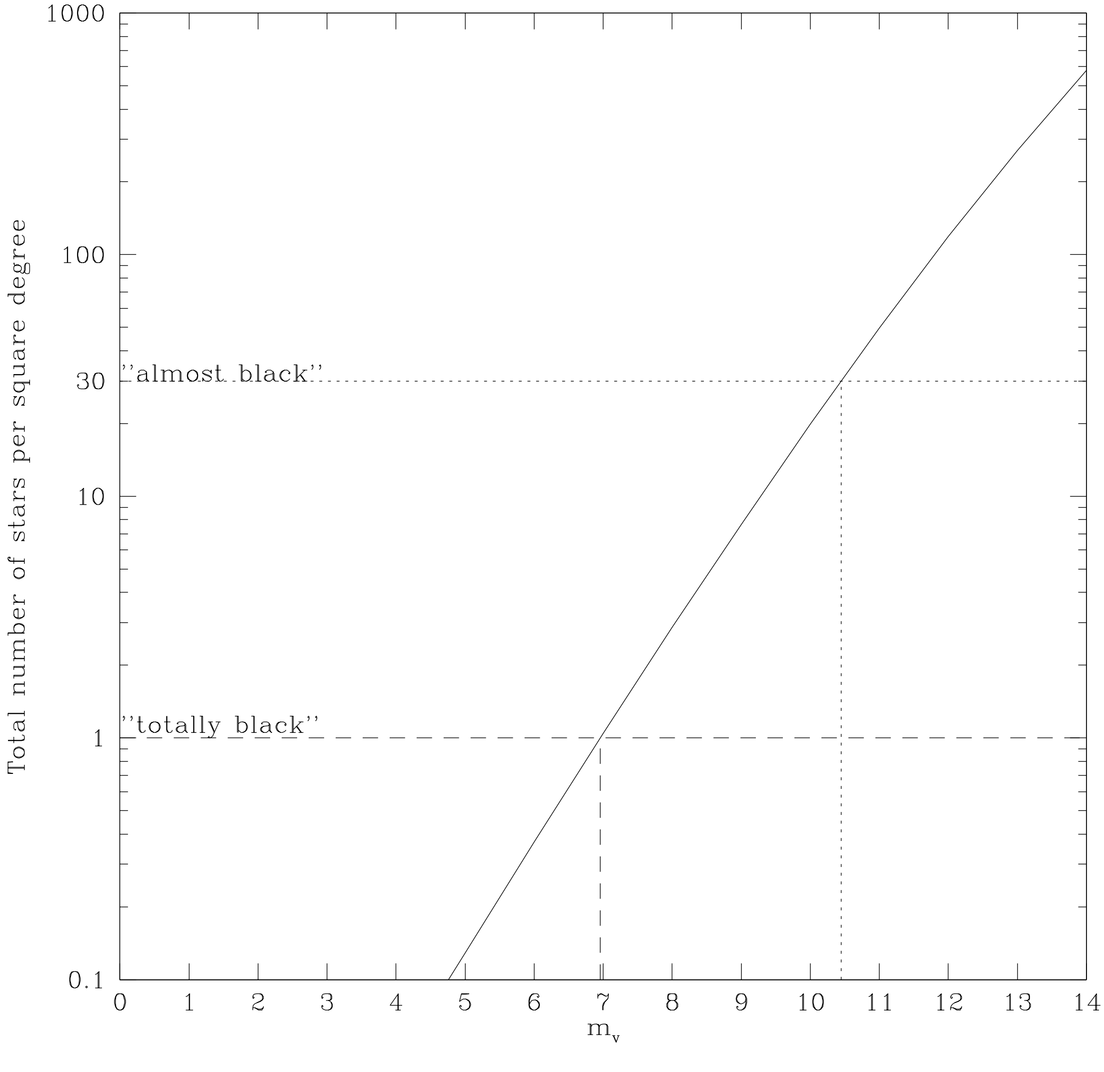
i. "almost black" regions:
Barnard's "almost black" have approximately 30 stars per square degree. Looking at Fig. 5, we see 30 stars per square degree up to a limiting magnitude of about 10.4. Thus the extra extinction is

$$A_v = 14 - 10.4 = \boxed{3.6\text{ mag.}}$$

ii. "totally black" regions:
Let's say the cutoff for a "totally black" region is 1 star per square degree. Then Fig. 5 indicates that a limiting magnitude of about 7 is appropriate. Thus

$$A_v = 14 - 7 = \boxed{7\text{ mag.}}$$

Figure 5:



1000
100
30
10
1
0.1
''almost black''
''totally black''
Total number of stars per square degree
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
m_v

Appendix

What is a solid angle of 1 square degree ($1\square^\circ$)? This is this the solid angle encompassed by taking 1 degree in one direction by 1 degree in the perpendicular direction. In other words,

$$1\square^\circ = \int d\Omega = \int_0^{1^\circ} \int_0^{1^\circ} d\theta d\phi. \tag{28}$$

If you simply integrate over the solid angle

$$\int d\Omega = \int_0^{1^\circ} \int_0^{1^\circ} \sin\theta \, d\theta d\phi$$

you will get the wrong answer. This solid angle is shown in Fig. 6.

Figure 6:

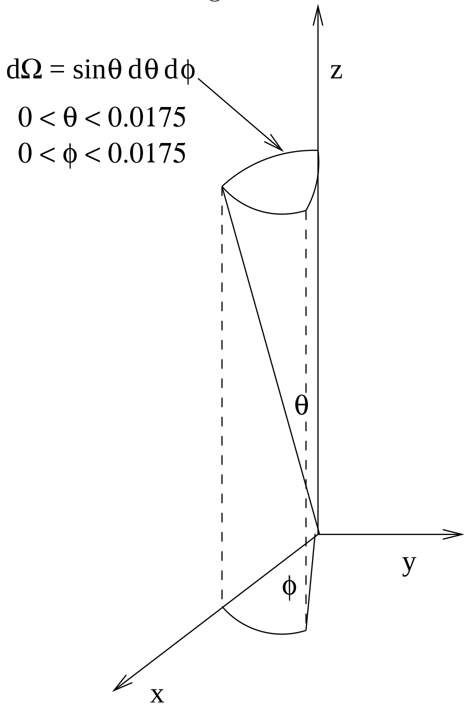
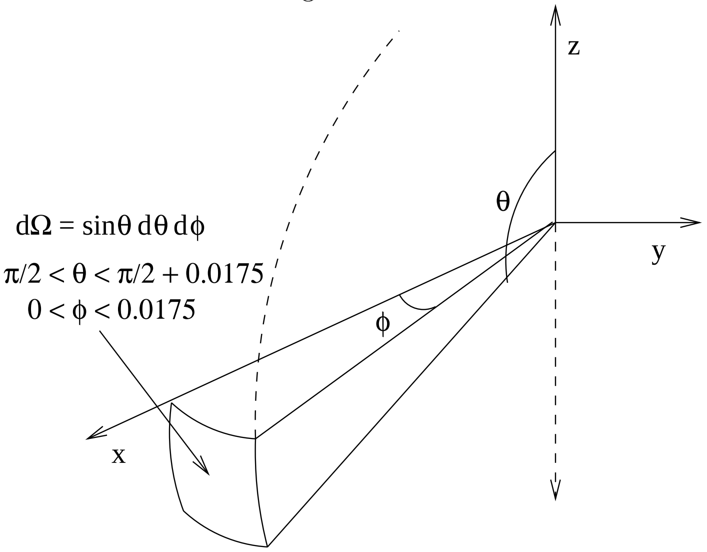


Figure 7:



The correct limits to use are

$$\int d\Omega = \int_0^{1^\circ} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+1^\circ} \sin\theta \, d\theta d\phi$$

which is shown in Fig. 7. Converting degrees to radians, $1^\circ = \pi/180 = 0.0175$ radians. Now $\sin(\frac{\pi}{2} + \frac{\pi}{180}) \approx \sin\frac{\pi}{2} = 1$. So, we can rewrite the solid angle as

$$\int d\Omega = \int_0^{1^\circ} \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+1^\circ} d\theta d\phi.$$

Shifting the limits on θ by $\pi/2$, we see that this is the same as Equation (28).

References

[1] Binney, J., & Merrifield, M. 1998, Galactic Astronomy (Princeton: Princeton University Press)

[2] Miller, G.E., & Scalo, J.M. 1979, ApJ, 41, 513