4. Classical Dynamics

4.1 Newtonian Gravity

Two point masses M_1 and M_2 positioned at $\mathbf{r}_1$ and $\mathbf{r}_2$ attract one another. M_1 feels a force from M_2.

$$\mathbf{F}_{12} = \frac{-GM_1M_2}{r_{12}^3}\mathbf{r}_{12}$$

where $r_{12} = \mathbf{r}_1 - \mathbf{r}_2$ is the vector from point 2 to point 1 and $r_{12} = |\mathbf{r}_{12}|$. M_2 feels a force from M_1

$$\mathbf{F}_{21} = \frac{-GM_1M_2}{r_{12}^3}\mathbf{r}_{21} = \frac{GM_1M_2}{r_{12}^3}\mathbf{r}_{12} = -\mathbf{F}_{12}$$

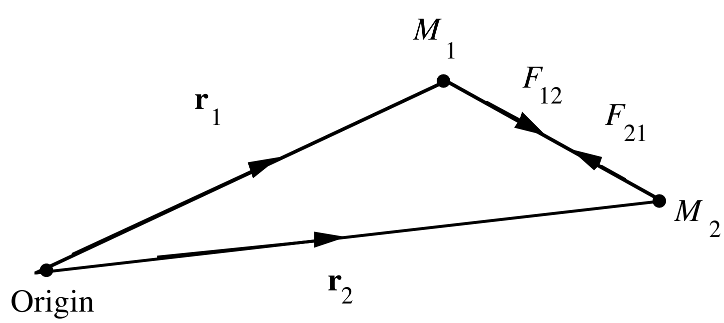


Fig . 4-1

Gravitational forces are equal and opposite in accord with Newton’s third law.

4.1.1 Newton's laws

1. A body remains at rest or in uniform motion unless acted on by a force
2. Force is equal to the time rate of change of momentum.
3. Action and reaction are equal in magnitudes and directly opposite in direction.

4.1.2 Gravitational potential

Gravitational forces can be represented by the gradient of a potential.

If $\mathbf{F}_{12}$ is the force on particle M_1 at $\mathbf{r}_1$ due to M_2 at $\mathbf{r}_2$

$$\mathbf{F}_{12} = -M_1 \nabla_1 V(\mathbf{r}_1)$$

where

$$V(\mathbf{r}_1) = \frac{-GM_2}{|\mathbf{r}_1 - \mathbf{r}_2|}$$

is the gravitational potential at $\mathbf{r}_1$ due to the presence of M_2 at $\mathbf{r}_2$ and

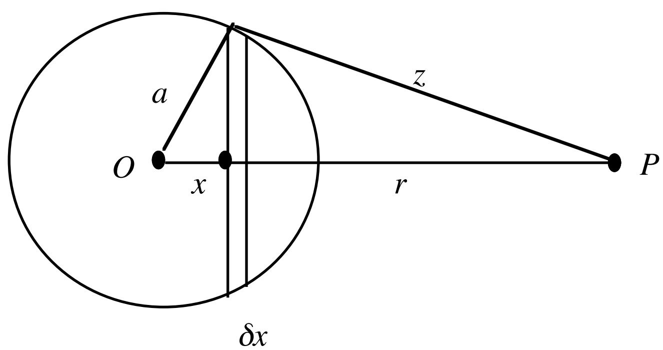
$$\nabla_1 = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial z_1} \right) .$$

If there are several masses M_i at points $\mathbf{r}_i$, the potential at any point $\mathbf{r}$ is

$$V(\mathbf{r}) = -\sum_{i=1}^{N} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}$$

where the sum excludes any mass at $\mathbf{r}$.

4.1.3 Gravitational attraction of a spherical shell



Consider a thin spherical shell of mass M and radius a. P is a particle of mass m at a distance r from the center O of the shell and outside it. Divide the shell into rings x, $x + \delta x$ by planes perpendicular to OP.

Area of ring is $2\pi a \sin\theta \times a\delta\theta = 2\pi a\,\delta x$

where $x = a\cos\theta$, $\delta x = a\sin\theta\,\delta\theta$

Surface area of the shell is $4\pi a^2$.

So mass of ring is

$$2\pi a \delta x \, \frac{M}{4\pi a^2} = \frac{M\delta x}{2a}$$

Every point of the ring is equidistant by z from P so gravitational potential at P due to the ring is

$$\frac{-G}{z} \, \frac{M\delta x}{2a} \quad .$$

Total for the shell is

$$\mathrm{V} = -\int_{-a}^{a} \frac{GM}{2az} \delta x \, .$$

Now $z^2 = a^2 + r^2 - 2rx$

$2zdz = - 2rdx$

$$\mathrm{V} = \int_{r+a}^{r-a} \frac{GM}{2ar} dz$$

$$= \frac{GM}{2ar} [(r-a) - (r+a)]$$

$$= -\frac{GM}{r} \quad .$$

Force on the particle is the same as that exerted by a particle of mass equal to that of the spherical shell placed at the center of the shell.

If P is inside the shell,

$$V = \int_{a+r}^{a-r} \frac{GM}{2ar}\,dz$$

$$= -\frac{GM}{a}$$

which is a constant. So the force $\dfrac{-dV}{dr}$ vanishes—the shell exerts no force on particles inside it.

Ostlie and Carroll (pp. 36-38) give a similar discussion but use the force rather than the potential.

4.1.4 Solid sphere

Suppose sphere is a solid with a mass distribution that is a function of r (or a constant). Add up the potentials of all the spherical shells—result is the same—gravitational potential on a particle outside a solid sphere is the same as that exerted by a particle of the mass of the sphere situated at its center.

Suppose particle is inside the solid sphere of mass M and radius R at a radius r. Shells with radii greater than r exert no force. Inside we have a solid sphere of mass Mr^3/R^3 so gravitational force on the particle is

$$G\frac{m}{r^2}\,\frac{Mr^3}{R^3} = \frac{GmMr}{R^3} \quad .$$

4.1.5 Two solid spherical bodies

Force on each particle of sphere *A* is the same as produced by a particle of mass m_B at the center of *B*. Add for all the particles. Gravitational force of *A* or *B* is the same as if the masses were particles at the centers of the two spheres.

4.2 The Two-body Problem

Equations of motion of two bodies at $\mathbf{r}_1'$ and $\mathbf{r}_2'$ with constant masses

$$M_1\ddot{\mathbf{r}}_1' = \frac{-GM_1M_2}{|\mathbf{r}_1' - \mathbf{r}_2'|^3}(\mathbf{r}_1' - \mathbf{r}_2')$$

$$M_2\ddot{\mathbf{r}}_2' = \frac{-GM_1M_2}{|\mathbf{r}_1' - \mathbf{r}_2'|^3}(\mathbf{r}_2' - \mathbf{r}_1') \quad .$$

Add the two equations

$$M_1\ddot{\mathbf{r}}_1' + M_2\ddot{\mathbf{r}}_2' = 0$$

(linear momentum is conserved).

The center of mass is at the position $\mathbf{R}$ where

$$\mathbf{R} = \frac{M_1 \mathbf{r}_1{}' + M_2 \mathbf{r}_2{}'}{(M_1 + M_2)}$$

Then $\ddot{\mathbf{R}} = 0$, $\dot{\mathbf{R}}$ = constant = v_{cm} (there is no external force) — in the absence of an external force, center of mass moves with uniform velocity.

Introduce coordinates $\mathbf{r}_1 = \mathbf{r}_1{}' - \mathbf{R}$

$\mathbf{r}_2 = \mathbf{r}_2{}' - \mathbf{R}$

$\mathbf{R} = \mathbf{R}_0 + \mathrm{v}_{cm} t$ (t is the time)

relative to the center of mass

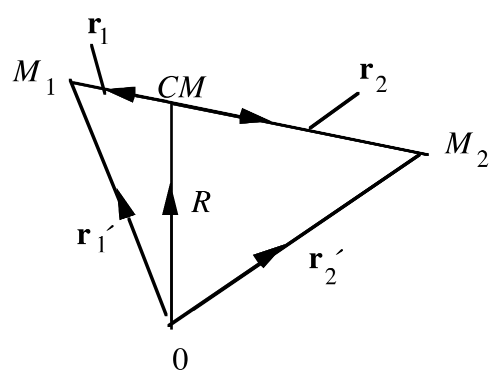


Fig. 4-2

Then

$$\mathbf{M}_1 \ddot{\mathbf{r}}_1 = \frac{-GM_1 M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2)$$

$$\mathbf{M}_2 \ddot{\mathbf{r}}_2 = \frac{-GM_1 M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_2 - \mathbf{r}_1) \quad .$$

This independence of origin and velocity is called Galilean invariance. So choose origin as the position of the center of mass—i.e. take $\mathbf{R} = 0$. (Then $\mathbf{r}_1 = \mathbf{r}_1{}'$ and $\mathbf{r}_2 = \mathbf{r}_2{}'$.) Now calculate the total angular momentum about origin.

$$\mathbf{L} = M_1 \mathbf{r}_1 \mathbf{x}\ \dot{\mathbf{r}}_1 + M_2 \mathbf{r}_2 \mathbf{x}\ \dot{\mathbf{r}}_2 \quad .$$

For any central force

$$M_1 \ddot{\mathbf{r}}_1 = -\lambda (\mathbf{r}_1 - \mathbf{r}_2)$$

$$M_2 \ddot{\mathbf{r}}_2 = -\lambda (\mathbf{r}_2 - \mathbf{r}_1)$$

where for gravitation $\lambda = \dfrac{GM_1 M_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$.

Then

$$\frac{d\mathbf{L}}{dt} = M_1\left(\dot{\mathbf{r}}_1 \text{ x } \dot{\mathbf{r}}_1 + \mathbf{r}_1 \text{ x } \ddot{\mathbf{r}}_1\right) + M_2\left(\dot{\mathbf{r}}_2 \text{ x } \dot{\mathbf{r}}_2 + \mathbf{r}_2 \text{ x } \ddot{\mathbf{r}}_2\right)$$
$$= \left(\dot{\mathbf{r}}_1 \text{ x } M_1\ddot{\mathbf{r}}_1\right) + \left(\mathbf{r}_2 \text{ x } M_2\ddot{\mathbf{r}}_2\right)$$
$$= -\lambda\left[\mathbf{r}_1 \text{ x } \left(\mathbf{r}_1 - \mathbf{r}_2\right) + \mathbf{r}_2 \text{ x } \left(\mathbf{r}_2 - \mathbf{r}_1\right)\right]$$
$$= -\lambda\left[\mathbf{r}_1 \text{ x } \mathbf{r}_2 + \mathbf{r}_2 \text{ x } \mathbf{r}_1\right] = 0 \ .$$

$\mathbf{L}$ = constant vector, perpendicular to the plane of the motion.

We can also prove that the total energy $E = T + \text{V}$ is constant where T is the kinetic energy and V the potential energy. Write the equations as

$$\text{T} = \frac{1}{2} M_1 \ \dot{\mathbf{r}}_1{}^2 + \frac{1}{2} M_2 \ \dot{\mathbf{r}}_2{}^2$$

$$M_1\ddot{\mathbf{r}}_1 = -\nabla_1 \text{V} , \ M_2\ddot{\mathbf{r}}_2 = -\nabla_2 \text{V} ,$$

and V as the potential

$$\text{V} = \frac{-GM_1M_2}{\left|\mathbf{r}_1 - \mathbf{r}_2\right|} \ .$$

Now

$$\frac{\text{dV}}{\text{dt}} = \nabla_1 \text{V} \cdot \dot{\mathbf{r}}_1 + \nabla_2 \text{V} \cdot \dot{\mathbf{r}}_2 \ .$$

Multiply equations of motion by $\dot{\mathbf{r}}_1$ to give

$$M_1 \ddot{\mathbf{r}}_1 \cdot \dot{\mathbf{r}}_1 = -\nabla_1 \mathbf{V} \cdot \dot{\mathbf{r}}_1$$

and by $\dot{\mathbf{r}}_2$ to give

$$M_2 \ddot{\mathbf{r}}_2 \cdot \dot{\mathbf{r}}_2 = -\nabla_2 \mathbf{V} \cdot \dot{\mathbf{r}}_2 \quad .$$

Thus

$$\frac{d}{dt}\left[\frac{1}{2}M_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}M_2\dot{\mathbf{r}}_2^2 + \mathrm{V}\right] = 0$$

i.e. $E = T + \mathrm{V} =$ constant.

To describe a two-body system we need six functions of time $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ and twelve constants (say initial positions and momenta) must appear in the solutions.

We now have determined 3 for $\mathbf{R}$, 3 for $\square\mathbf{V}_{cm}$, 3 for $\mathbf{L}$ and 1 for E (these also apply to a many-body system). The two remaining are valid for two-body systems only.

4.2.1 Two-body orbits

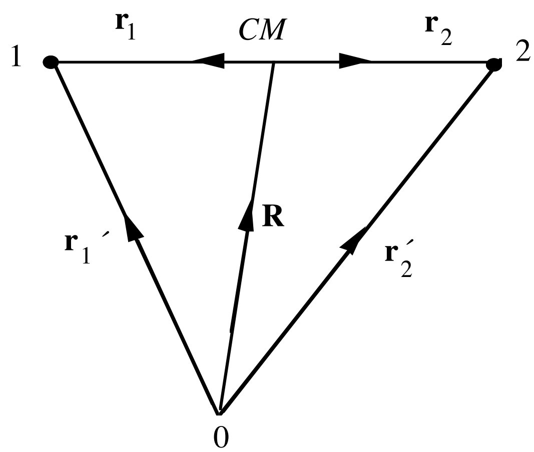


Fig. 4-3

$$\mathbf{R} = \frac{M_1\mathbf{r}_1{}' + M_2\mathbf{r}_2{}'}{M} \quad , \; M = M_1 + M_2$$

Introduce $\mathbf{r} = \left(\mathbf{r}_1{}' - \mathbf{r}_2{}'\right)$

Then $\mathbf{r}_1 = \dfrac{M_2\mathbf{r}}{M}$, $\mathbf{r}_2 = \dfrac{-M_1}{M}\mathbf{r}$, $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$

$$\mathbf{r}_1{}' = \mathbf{r}_1 + \mathbf{R} \quad , \mathbf{r}_2{}' = \mathbf{r}_2 + \mathbf{R} \quad .$$

Equation of motion of 1 and 2 are

$$M_1 \ddot{\mathbf{r}}_1{}' = - \frac{GM_1 M_2 \, \mathbf{r}}{r^3}$$

$$M_2 \ddot{\mathbf{r}}_2{}' = + \frac{GM_1 M_2 \, \mathbf{r}}{r^3} \quad .$$

Now consider the *relative* motion of the two particles

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1{}' - \ddot{\mathbf{r}}_2{}'$$

$$= - \frac{GM_2 \, \mathbf{r}}{r^3} - \frac{GM_1 \, \mathbf{r}}{r^3}$$

$$= - \frac{GM \, \mathbf{r}}{r^3} \quad .$$

This can be written as force = mass × acceleration

$$\frac{M_1 M_2}{M} \ddot{\mathbf{r}} = - \frac{GM_1 M_2 \, \mathbf{r}}{r^3} \quad .$$

Define a *reduced mass* μ.

$$\mu = \frac{M_1 M_2}{M_1 + M_2} \quad , \quad \frac{1}{\mu} = \frac{1}{M_1} + \frac{1}{M_2} \quad .$$

Then

$$\mu \ddot{\mathbf{r}} = -\frac{G\mu M \mathbf{r}}{r^3} \quad .$$

The equation describes the motion of a fictitious particle of mass μ under the gravitational attraction of a particle of mass $M = M_1 + M_2$ separated by $\mathbf{r}$. The force can be written as the gradient with respect to $\mathbf{r}$ of a potential

$$V(r) = -\frac{GM_1 M_2}{r} = \frac{-G\mu M}{r} \quad .$$

The total angular momentum of the system about the center of mass is

$$\mathbf{L} = \mathbf{r}_1 \times M_1 \dot{\mathbf{r}}_1 + \mathbf{r}_2 \times M_2 \dot{\mathbf{r}}_2 \quad .$$

With

$$\mathbf{r}_1 = \frac{M_2 \mathbf{r}}{M} \;, \; \mathbf{r}_2 = -\frac{M_1 \mathbf{r}}{M}$$

$$\mathbf{L} = \frac{M_1 M_2}{M} \mathbf{r} \times \left(\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 \right)$$

$$= \mu \mathbf{r} \times \dot{\mathbf{r}} \, .$$

The center of mass moves in a straight line and makes no contribution to the angular momentum ($\mathbf{R} \times \mathbf{V}$cm = 0).

The angular momentum of the two-body system is identical to that of a single particle of mass μ moving with the relative velocities of the two bodies.

The kinetic energy of the two bodies is

$$T = \frac{1}{2} M_1 \dot{\mathbf{r}}_1{}'^2 + \frac{1}{2} M_2 \dot{\mathbf{r}}_2{}'^2$$

$$= \frac{1}{2} M_1 \left(\dot{\mathbf{r}}_1 + \dot{\mathbf{R}}\right)^2 + \frac{1}{2} M_2 \left(\dot{\mathbf{r}}_2 + \dot{\mathbf{R}}\right)^2$$

$$= \frac{1}{2} M_1 \left(\frac{M_2}{M}\right)^2 \dot{\mathbf{r}}^2 + \frac{1}{2} M_2 \left(\frac{M_1}{M}\right)^2 \dot{\mathbf{r}}^2$$

$$+ \dot{\mathbf{R}} \left(\frac{M_1 M_2}{M} \dot{\mathbf{r}} - \frac{M_2 M_1}{M} \dot{\mathbf{r}}\right)$$

$$+ \frac{1}{2} \left(M_1 + M_2\right) \dot{\mathbf{R}}^2$$

$$\therefore T = \frac{1}{2} \frac{M_1 M_2}{M} \dot{\mathbf{r}}^2 + \frac{1}{2} M \mathrm{v}_{\mathrm{cm}}{}^2$$

$$= \frac{1}{2} \mu \dot{\mathbf{r}}^2 + \frac{1}{2} M \mathrm{v}_{\mathbf{cm}}{}^2 .$$

The kinetic energy of the two-body system is the kinetic energy of relative motion of a particle of mass μ plus the kinetic energy of the total mass M moving with the velocity of the center of mass. The potential energy is

$$\mathrm{V} = -\frac{GM_1M_2}{r} = \frac{-G\mu M}{r} \quad .$$

so the total energy of the system is the total energy of the particle of mass μ moving in the field of a particle of mass M plus the kinetic energy of mass M moving with the center of mass

$$E = \frac{1}{2}\mu\dot{\mathbf{r}}^2 + \frac{1}{2}M\mathrm{v}_{\mathrm{cm}}^{\;2} - \frac{G\mu M}{r} \; .$$

In Cartesian coordinates

$$\mathbf{L} = (0,\, 0, L)\;,\; \mathbf{r} = (\mathrm{x,y},0),\; \dot{\mathbf{r}} = (\dot{x},\dot{y},0)$$

angular momentum equation

$$L = \mu(x\dot{y} - \mathrm{y}\dot{x})$$

energy equation

$$E = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2\right) - \frac{G\mu M}{\left(x^2 + y^2\right)^{\frac{1}{2}}} \quad .$$

Better in polar coordinates

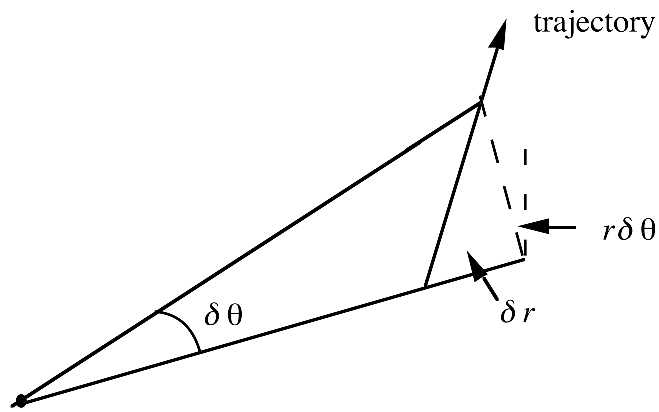


Fig. 4-4

or cylindrical coordinates (r, θ, z) with z perpendicular to the plane.

In time t,

$$\mathbf{r}(t) \to \mathbf{r}(t+\delta t)$$

$$\dot{\mathbf{r}} = \frac{\mathbf{r}\left(t+\delta t\right) - \mathbf{r}(t)}{\delta t}$$

$$= \left(\frac{\delta r}{\delta t}\ ,\ \frac{r\delta\theta}{\delta t}\ ,\ 0\right)$$

So components of the vector $\dot{\mathbf{r}} = \left(\dot{r}\ ,\ r\ \dot{\theta}\ ,\ 0\right)$

Then

$$\mathbf{r}\ \mathrm{x}\ \dot{\mathbf{r}} = \left(\ 0,\ 0,\ r^2\dot{\theta}\ \right)$$

$$\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} = \dot{r}^2 + r^2\ \dot{\theta}^2$$

So $L = \mu r^2 \dot{\theta}$ where $\dot{\theta}$ = angular velocity

$$E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{G\mu M}{r} \quad .$$

Angular momentum equation can be interpreted geometrically.

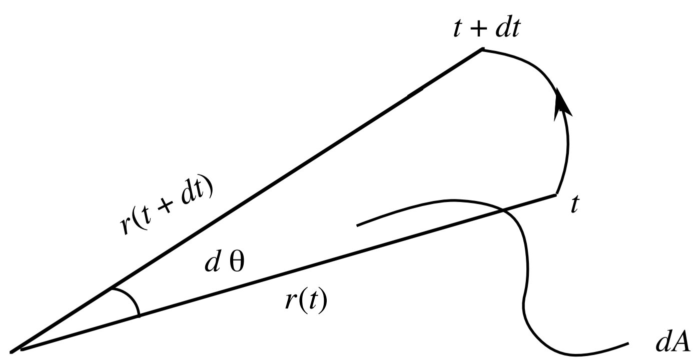


Fig. 4-5

In the interval t, $t + dt$, the orbit sweeps out an area dA and

$$dA = \frac{1}{2} r \text{ x } r d\theta$$

so

$$\frac{dA}{dt} = \frac{1}{2} r \text{ x } \frac{r d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

$$\therefore \quad \frac{dA}{dt} = \frac{L}{2\mu} = \text{constant} .$$

This is Kepler's second law. The radius vector to a planet sweeps out equal areas in equal intervals of time.

We can proceed further to obtain an equation in a single variable r. We have

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{1}{\mu}\right)^2 \frac{L^2}{r^2} \quad .$$

The last term is the centrifugal repulsion. It is an *inertial force* that arises to take accou[redacted] of the angu[redacted] motion. We are in effect using a rotating frame with respect to w[redacted]h the parti[redacted] has no angular motion and we need consider only radial moti[redacted] Consider a[redacted]rticle moving with constant angular velocity ω in a circle and ask [redacted]at is the ef[redacted]tive force. The velocity is $\omega \times \mathbf{r}$ and the acceleration is $\omega \times$ [redacted] $\omega \times \mathbf{r}) = r$ [redacted] ^{2}. So the force is $\mu r \dot{\theta}^2$ which is the centrifugal repulsion. The [redacted]responding [redacted]tential is $\frac{-1}{2} \mu r^2 \dot{\theta}^2 = -\frac{\mu}{2}\left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}$.

4.2.2 Runge-Lenz Vector

For a central force proportional to $1/r^2$, there is an additional conserved vector called the Runge-Lenz vector (though it was written down in 1799 by Laplace). The Runge-Lenz vector is

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu^2 GM \, \frac{\mathbf{r}}{r}$$

Then

$$\frac{d}{dt}(\mathbf{p} \times \mathbf{L}) = \dot{\mathbf{p}} \times \boldsymbol{L} = \mu \ddot{\mathbf{r}} \times \mathbf{L} = -\frac{\mu GM}{r^3} \, \mathbf{r} \times \mathbf{L}$$

$$= -\frac{\mu^2 GM}{r^3} \, \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}).$$

Use vector identity

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$$

$$\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \mathbf{r}(\mathbf{r} \cdot \dot{\mathbf{r}}) - \dot{\mathbf{r}} \, r^2$$

$$= r^3 \left\{ \mathbf{r} \, \frac{\mathbf{r} \cdot \dot{\mathbf{r}}}{r^3} - \frac{\dot{\mathbf{r}}}{r} \right\}.$$

But $\mathbf{r} \cdot \dot{\mathbf{r}} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}}(\mathbf{r} \cdot \mathbf{r}) = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{dt}}\left(r^2\right) = r\dot{r}$

and

$$\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = r^3 \left\{ \frac{1}{r^2}\, \dot{r}\; \mathbf{r} - \frac{\dot{\mathbf{r}}}{r} \right\}.$$

Now $\dfrac{d}{dt}$ (unit vector) $= \dfrac{d}{dt}\left(\dfrac{\mathbf{r}}{r}\right) = -\dfrac{1}{r^2}\,\dot{r}\;\mathbf{r} + \dot{\mathbf{r}}/r$.

So $$\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -r^3 \frac{\mathrm{d}}{\mathrm{dt}} \hat{\mathbf{r}}, \qquad \hat{\mathbf{r}} = \mathbf{r}/r .$$

$$\frac{d}{dt}(\mathbf{p} \times \mathrm{L}) = \frac{+\mu^2 GM}{r^3}\, r^3\, \frac{d\hat{\mathbf{r}}}{dt} = \mu^2 GM \frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

$$\mathbf{p} \times \mathbf{L} - \frac{\mu^2 GM\mathbf{r}}{r} = \text{constant} = \mathbf{A}, \text{ say}$$

Then $\mathbf{A} \cdot \mathbf{L} = 0$.

A is a fixed vector in the plane of the orbit.

4.2.3 Orbits

To find the trajectory as a function of time, we have to integrate the pair of equations with respect to time.

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2}$$

$$\left(\frac{dr}{dt}\right)^2 = \frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{1}{\mu}\right)^2 \frac{L^2}{r^2} \quad .$$

But we can find the shape of the orbit (which is r as a function of θ)

$$\frac{dr}{d\theta} = \frac{dr}{dt} / \frac{d\theta}{dt}$$

$$= \frac{\mu}{L} r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}\right]^{1/2}$$

or

$$d\theta = \frac{(L/\mu)\, dr}{r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}\right]^{1/2}} \quad .$$

Integrate

$$\int_{\theta_0}^{\theta} d\theta = \theta - \theta_0 = \int^{r} \frac{(L/\mu)\, dr}{r^2 \left[\frac{2E}{\mu} + \frac{2GM}{r} - \left(\frac{L}{\mu}\right)^2 \frac{1}{r^2}\right]^{1/2}}$$

where θ_0 is a constant of integration.

Define a scale length by

$$r_o = \frac{(L/\mu)^2}{GM} = \frac{L^2}{GM\mu^2}$$

and a second constant ε by

$$\varepsilon^2 = 1 + \frac{2(E/\mu)(L/\mu)^2}{(GM)^2}$$

$$= 1 + 2\mathrm{EL}^2/(GM)^2\mu^3 \quad .$$

For a bound orbit, $\varepsilon < 1$ and E is negative. Then the integral can be written

$$\int^r \frac{r_o dr}{r^2\left\{\varepsilon^2 - (1 - r_o/r)^2\right\}^{1/2}} = \theta - \theta_o \ .$$

Introduce a new variable u by

$$(1 - r_o/r)^2 = \varepsilon^2 \cos^2 u$$

or

$$\varepsilon \cos u = \pm(1 - r_o/r) \ .$$

We obtain

$$\int_o^u du = \theta - \theta_o$$

i.e. $u = \theta - \theta_o$, $\cos u = \cos(\theta - \theta_o)$

Hence

$$\varepsilon \cos(\theta - \theta_o) = 1 - r/r_o$$

or

$$\frac{1}{r} = \frac{1}{r_o}\left[1 \pm \varepsilon \cos\left(\theta - \theta_o\right)\right] .$$

We can choose ±, we choose +

$$\frac{1}{r} = \frac{1}{r_o}\left\{1 + \varepsilon \cos\left(\theta - \theta_o\right)\right\}$$

is the orbit equation. It is the equation of a conic section. ε is called the *eccentricity*. There are three possibilities.

$\underline{\varepsilon < 1}$ If $\varepsilon < 1$, r is always finite—the particles remain bound with

$$r_p = \frac{r_o}{1 + \varepsilon} \leq r \leq \frac{r_o}{1 - \varepsilon} = r_a$$

where r_p and r_a are the nearest and furthest parts of the trajectory from a focus, called respectively the perihelion and the aphelion. Then

$$\varepsilon = \frac{r_a - r_p}{r_a + r_p} \quad .$$

If $\varepsilon = 0$, the motion is a circle and the two foci coalesce at the center and $r_o = a$ is the radius.

Let us express the orbit in Cartesian coordinates.

$$x = r \cos (\theta - \theta_0)$$

$$y = r \sin (\theta - \theta_0) \ .$$

Then the orbit equation is

$$\frac{1}{\left(x^2 + y^2\right)^{1/2}} = \frac{1}{r_o} \left\{ 1 + \frac{\varepsilon x}{\left(x^2 + y^2\right)^{1/2}} \right\}$$

which is

$$\left(\frac{x + \varepsilon a}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

where $a = \dfrac{r_o}{1 - \varepsilon^2}$, $b = \dfrac{r_o}{\left(1 - \varepsilon^2\right)^{1/2}}$.

a is the *semi-major* axis, b is the *semi-minor* axis, r_o is the *semi-latus* rectum.

The axial ratio is $\frac{b}{a} = (1-\varepsilon^2)^{1/2}$

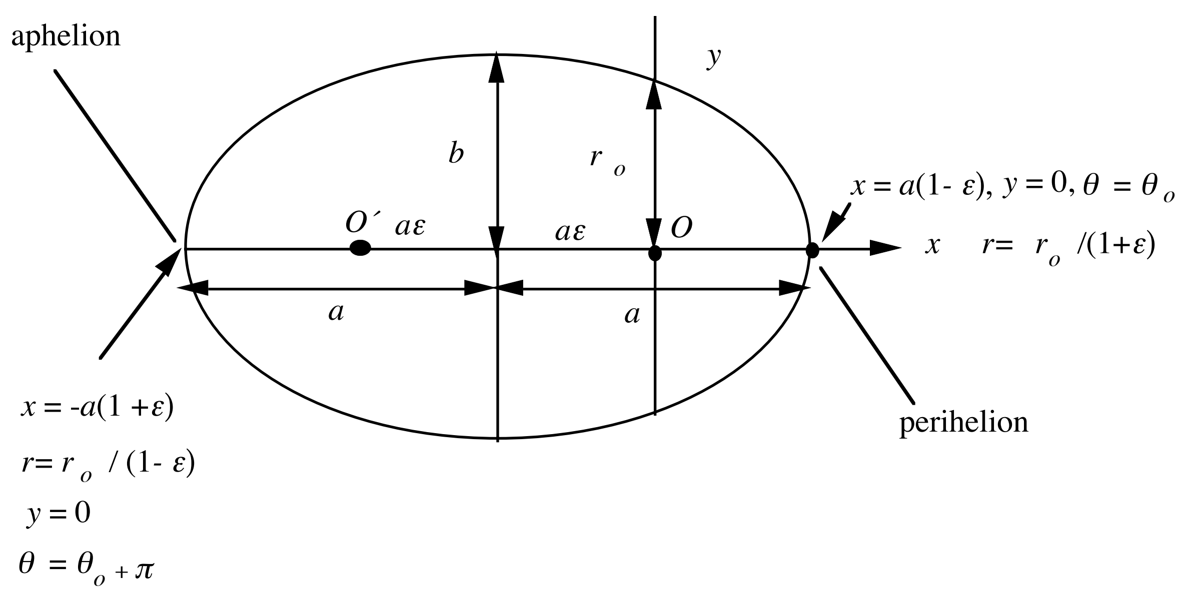


Fig. 4-6

The origin O of coordinates is the focus at the center of mass (close to the Sun). If the origin is taken at the center (the midpoint of the two foci) equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

The standard definition (and how you can draw it) is the locus of the point P such that $r + r' =$ constant where r and r' are the distances from points O and O'.

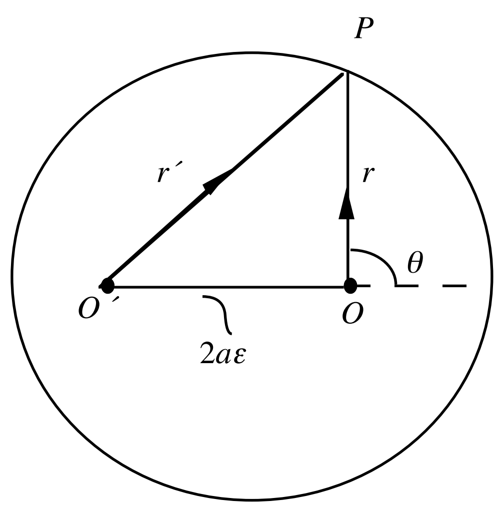


fig. 4-7

$$r'^2 = r^2 + (2a\varepsilon)^2 + 4a\varepsilon r \cos\theta$$

Orbit equation:

$$r = a(1\text{-}\varepsilon)^2/(1 + \varepsilon \cos\,\theta)$$

$$\therefore\ \varepsilon r \cos\,\theta = (1\text{-}\varepsilon)^2 \text{-}\ r$$

$$r'^2 = r^2 + 4a^2\varepsilon^2 + 4a^2(1\text{-}\varepsilon^2)\text{-}4ar$$

$$= (\mathrm{r}\text{-}2a)^2$$

$$\therefore\ r + r' = 2a.$$

$\varepsilon < 1$ implies $E < 0$, the total energy of a bound system is negative.

The orbit is periodic and closed.

The period is obtained from $dt/d\theta = (\,d\theta/\,dt)^{-1}$,

$$\frac{d\theta}{dt} = \frac{(L/\mu)}{r^2}$$

using $\frac{1}{r} = \frac{1}{r_o}\left[1 + \varepsilon\cos\left(\theta - \theta_o\right)\right]$.

Hence $\int^{t} dt = \frac{r_o^2}{(L/\mu)} \int \frac{d\theta}{\left[1 + \varepsilon\cos\left(\theta - \theta_o\right)\right]^2}$.

In going around the orbit, $\theta \rightarrow \theta + 2\pi$

$$t \rightarrow t + \tau$$

where τ is the period so

$$\frac{r_o^2}{(L/\mu)} \int_0^{2\pi} \frac{d\theta}{\left[1 + \varepsilon \cos \theta\right]^2} = \tau.$$

Use the substitution $t = \tan(\theta/2)$

$$d\theta = \frac{2dt}{1 + t^2} \text{ , } \cos \theta = \frac{1 - t^2}{1 + t^2} \quad .$$

Then

$$\tau = \frac{r_o^2}{(L/\mu)} \cdot \frac{2\pi}{\left(1 - \varepsilon^2\right)^{3/2}}$$

$$\text{or} \quad \tau = \frac{2\pi a^{3/2}}{(GM)^{1/2}} \quad .$$

The points at which the velocities are at right angles to the radius vector are called *apses*. The apse nearer to the Sun is the perihelion and the point further away is the aphelion (Fig. 4-6).

An alternative proof: integrate $\frac{dA}{dT}$ over the period P-A is the area of the orbit. Kepler's second law is

$$\frac{dA}{dT} = \frac{L}{2u} \qquad \text{(cf. } 4-18\text{)}$$

So $A = \dfrac{LP}{2\mu}$ is the area of the ellipse.

Thus $A = \pi ab$.

Use $\dfrac{b^2}{a} = r_o = L^2 / GM\mu^2$.

Then $P^2 = \dfrac{4\mu^2 A^2}{L^2} = \dfrac{4\mu^2}{L^2}\pi^2\ a^2\ a\ \dfrac{L^2}{GM\mu^2}$

$$= \frac{4\pi^2}{GM}\ a^3.$$

At perihelion r_p and aphelion r_a,

$$L = \mu \mathsf{V} r = \mu \mathsf{V}_p\ a\ (1\text{-}\varepsilon) = \mu \mathsf{V}_a\ a\ (1\text{+}\varepsilon)$$

$$\frac{\mathsf{V}_p}{\mathsf{V}_a} = \frac{(1+\varepsilon)}{(1-\varepsilon)}$$

$$E = \frac{\mu \mathsf{V}_p^{\ 2}}{2} - \frac{GM\mu}{r_p} = \frac{\mu \mathsf{V}_a^{\ 2}}{2} - \frac{GM\mu}{r_a} \quad .$$

Replace V_p by V_a $(1+\varepsilon)/(1-\varepsilon)$ to get

$$\mathsf{V}_a = \sqrt{\frac{GM(1-\varepsilon)}{a(1+\varepsilon)}}$$

$$E = \frac{\mu GM(1-\varepsilon)}{2a(1+\varepsilon)} - \frac{\mu GM}{a(1+\varepsilon)}$$

$$= -\frac{GM\mu}{2a} \quad .$$

So $a = -\dfrac{GM_1 M_2}{2E}$ depends only on energy.

ε depends on energy and angular momentum.

Kepler's Laws: for bound orbits,

1. the planets move in ellipses with the center of mass (the Sun) at one focus.

2. A line from the Sun sweeps out equal areas in equal times

$$\frac{dA}{dt} = \frac{1}{2}(L/\mu) \quad .$$

(*A* is the area here, not the magnitude of the Runge Lenz vector). $\frac{dA}{dt}$ does not depend on ε so the law applies also to unbound orbits with $\varepsilon \geq 1$.

3. The square of the period of revolution is proportional to the cube of the semi-major axis

$$\tau^2 = \frac{4\pi^2}{GM} a^3 \quad .$$

If we ignore the mass of the planet compared to the mass of the Sun, $M = M_\odot$. Then, if τ is measured in years, call it the period *P*, and *a* is measured in AU,

$$P^2 = a^3 \; .$$

More generally, for a total mass M measured in $M_\odot$

$$P\,(\text{years})^2 = \frac{a\,(AU)^3}{M\,(M_\odot)} \quad .$$

Mean angular velocity $\omega = 2\pi/\tau$ so

$$\omega^2 = GM/a^3 \quad .$$

To determine velocity at r, use conservation of energy

$$E = \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{G\mu M}{r} = \frac{-G\mu M}{2a}$$

$$= \frac{1}{2}\mu\dot{\mathbf{r}}^2 - \frac{G\mu M}{r} \quad .$$

The relative motion is an ellipse. The actual bodies move in ellipses of the same shape but different sizes and all have the same angular velocity. Thus

$$\mathbf{r}_1 = \frac{M_2}{M}\mathbf{r}\ ,\ \dot{\mathbf{r}}_1 = \frac{M_2}{M}\dot{\mathbf{r}}$$

$$\mathbf{r}_2 = -\frac{M_1\mathbf{r}}{M}\ ,\ \dot{\mathbf{r}}_2 = -\frac{M_1\dot{\mathbf{r}}}{M} \quad .$$

The Sun moves in a small orbit around the center of mass and the planet in a large orbit around the center of mass, always positioned so that they are on opposite sides of the center of mass.

Suppose we ignore the other planets and consider only Jupiter. The average Jupiter-Sun distance is 5.2 AU. The mass ratio of Jupiter to the Sun is 0.95×10^{-3}. The radius of the Sun's orbit is

$$\frac{M_1 a}{M} = 0.95 \times 10^{-3} \times 5.2 \times 1.5 \times 10^{8} \text{ km}$$

$$= 7.4 \times 10^{5} \text{ km} .$$

The radius of the Sun is comparable at $R_\odot = 6.696 \times 10^5$ km.

<u>$\varepsilon > 1$</u> $\varepsilon = 1 + 2EL^2/(GM)^2 \mu^3$ $E > 0$ and orbit is unbound

- a hyperbola.

Put $\frac{1}{r} = \frac{1}{r_o}\left[1 + \varepsilon \cos(\theta - \theta_o)\right]$ into Cartesian coordinates

$$\frac{1}{\left(x^2 + y^2\right)^{1/2}} = \frac{1}{r_o}\left(1 + \frac{\varepsilon x}{\left(x^2 + y^2\right)^{1/2}}\right)$$

$$\therefore \quad \frac{(x - \varepsilon a)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\text{where } a = \frac{r_o}{\varepsilon^2 - 1} , \; b = \frac{r_o}{\left(\varepsilon^2 - 1\right)^{1/2}}$$

$$\frac{b}{a} = \left(\varepsilon^2 - 1\right)^{1/2} .$$

With origin O at the midpoint, equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

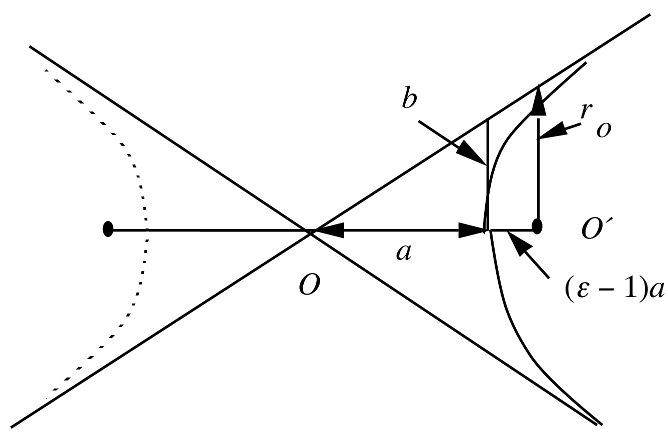


Fig. 4-8

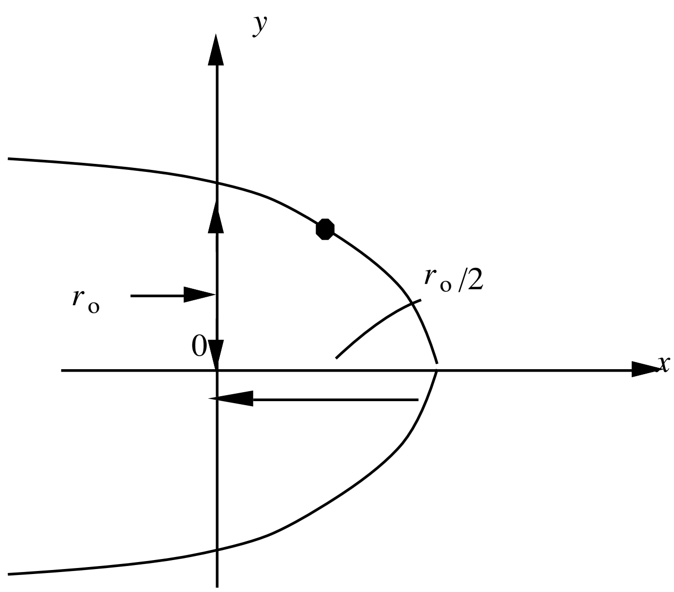
$\underline{\varepsilon = 1}$ $E = 0$, total energy is zero

Orbit is a parabola

$$y^2 = r_o^2 - 2r_o x$$

For a given distance r, $E = 0$ defines the escape velocity $1/2\ \mu V^2_{esc} = \dfrac{GM\mu}{r}$.

If $v > V_{esc}$, the particle escapes the gravitational field of M.



Parabola - single pass orbit

Fig. 4-9

4.2.4 Mass of Sun

The sidereal period of a planet, τ, (denoted earlier by P), is related to the semi-major axis

$$\tau^2 = \left(\frac{4\pi^2}{GM}\right)a^3$$

where

$$M = M_{\odot} + M_{Planet}$$

Measurements of τ

Planet	τ (days)	a (AU)	GM (10^{26} cm^3 s^{-1})	ε
Mercury	87.969	0.387099	1.32714	0.206
Venus	224.701	0.723332	1.32713	0.007
Earth	365.256	1.000000	1.32713	0.017
Mars	686.980	1.523691	1.32712	0.093
Jupiter	4332.589	5.202803	1.32839	0.048
Saturn	10759.22	9.53884	1.32750	0.056
Uranus	30685.4	19.1819	1.32715	0.047
Neptune	60189	30.0578	1.32723	0.009
Pluto	90465	39.44	1.32727	0.249

Given τ and a, we can obtain $M_p + M_\odot$.

From low mass planets

$$G\,M_\odot = 1.32713 \times 10^{26}\ \mathrm{cm}^3\ \mathrm{s}^{-2}.$$

G itself is known only to 1 part in 10^4.

$$G = 6.670 \pm 0.004\ 10^{-8}\ \mathrm{cm}^3\ \mathrm{s}^{-2}\ \mathrm{g}^{-1}\ .$$

Then $M_\odot = 1.989 \pm 0.001\ 10^{33}$g.

We can also derive mass of Jupiter.

$$G(M_\odot + M_J) = 1.32839\ 10^{26}\ \text{cm}^3\ \text{s}^{-1}$$

$$G(M_\odot + M_E) = 1.32713\ 10^{26}\ \text{cm}^3\ \text{s}^{-1}$$

$$\therefore\ G(M_J - M_E) = 1.26\ 10^{23}\ \text{cm}^3\ \text{s}^{-1}$$

$$\frac{M_J - M_E}{M_\odot} = 0.000949\ .$$

$$M_J \sim 0.000949 \text{ x } M_\odot$$

$$= 1.89 \text{ x } 10^{30}\ \text{g}$$

Better estimates can be made from the orbits of planetary satellites and spacecraft.

4.2.5 Interplanetary travel

Spacecrafts travel in orbits around the Sun. Suppose a spacecraft is directed to Mercury. We wish to place it in an orbit around the Sun that is tangent to the Earth at aphelion and tangent to Mercury at perihelion.

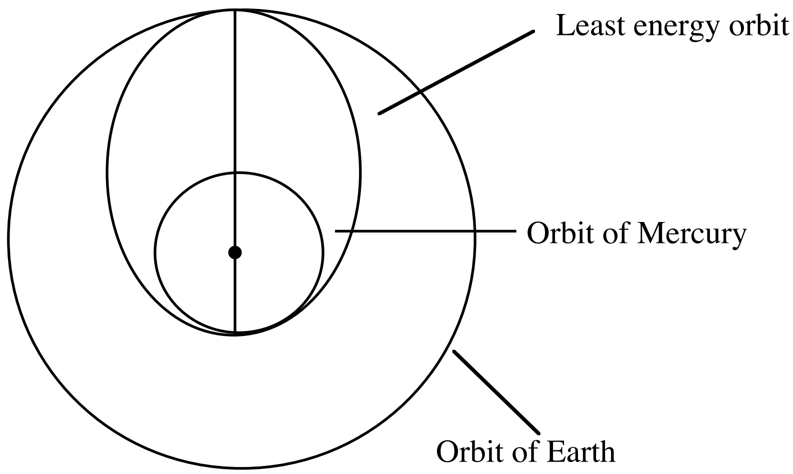


Fig. 4-10

That orbit has the smallest a and therefore takes the least energy. Assume orbits of Earth and Mercury are circular. Major axis is the sum of the aphelion and perihelion distances

$$2a = 0.387 + 1.000 = 1.387 \text{ AU}$$

$$a = 0.694 \text{ AU} = 1.04 \times 10^{11} \text{ m.}$$

The initial orbital speed at aphelion comes from the conservation of energy

$$V_a^2 = GM_\odot\left(\frac{2}{r_a} - \frac{1}{a}\right)$$

(We are ignoring the gravitational fields of the Earth and Mercury) Then with $r = 1.496\times 10^{11}$ m,

$$M_\odot = 1.989 \times 10^{30} \text{ kg}$$

$$V_a = 22 \text{ km s}^{-1} .$$

Each is orbiting the Sun at 30 km s^{-1} so we launch at 8 km s^{-1} in a direction *opposite* to the direction of the Earth's motion.

4.2.6 Moment of intertia of a spinning sphere

The angular momentum of a particle of mass m orbiting about a center with angular velocity ω is

$$L = mr^2\dot{\theta} = mr^2\omega = I\omega$$

and its rotational kinetic energy is

$$T = \frac{1}{2}mr^2\dot{\theta}^2 = \frac{1}{2} I\omega^2 .$$

The angular momentum of a spherical body rotating about an axis with angular velocity ω is similarly $I\omega$ and the kinetic energy is $\frac{1}{2} I\omega^2$ where I is called the moment of inertia. For a uniform sphere of mass M and radius R,

$$I = \int\int\int \rho r^2 \left(x^2 + y^2\right) dr d\Omega$$

$$= \frac{8\pi}{15} \rho R^5 = \frac{2}{5} MR^2 \quad .$$

4.3 Binary stars

More than half the stars are multiple stars, most of which are binary pairs. For solar-type stars, the observed ratios of single: double: triple: quadrupole systems is 45:46:8:1. There are several classes of binaries:

visual binaries: both can be detected orbiting in ellipses about one another (Sirius is a famous example - Sirius A is a main sequence star of spectral type A1, Sirius B is a white dwarf of spectra type A5. The period is 49.9 years.

Sirius was first discovered as an **astrometric binary**.

Astrometric binaries are binaries in which only one star is observed but its motion is oscillatory, indicating the perturbing presence of a dim companion.

Spectroscopic binaries are visually unresolved but periodic oscillations occur in their spectrum. If only one stellar spectrum is observed, the binary is *single-lined*; if both are observed, the binary is double-lined.

Eclipsing binaries occur when the two stars eclipse one another, producing periodic changes in apparent brightness.

Periods of binary stars vary from a few hours to hundreds of years. From data on the periods we can use the law of gravitation to infer masses. Consider a double-lined spectroscopic binary. The spectra of the two stars are superimposed. We can use Doppler shifts to measure the radial velocities of each star, though they may be too close for their orbits to be distinguished.

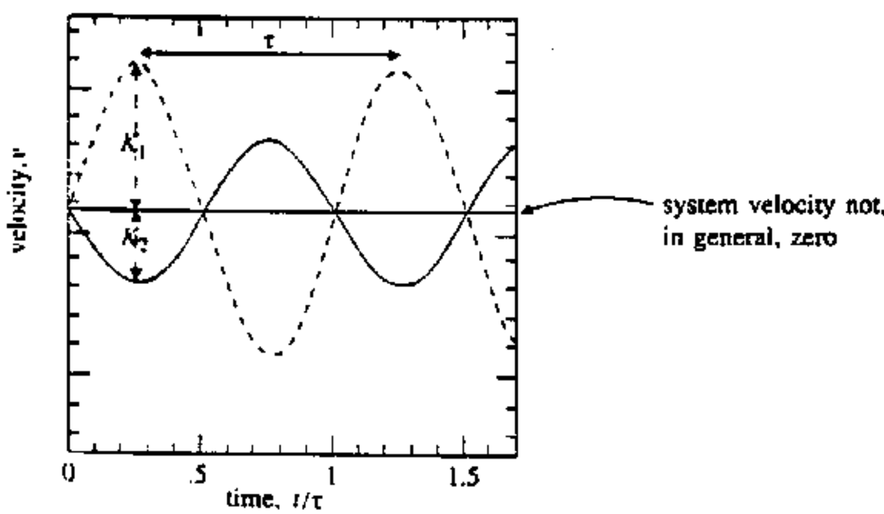


Fig. 4-10

The line joining the stars is rotating with angular velocity Ω and $\mathbf{K}_1$ and $\mathbf{K}_2$ are the inferred radial velocities.

The figure shows the individual radial velocities. From it we obtain the peak velocities of each star and the binary period τ. If the shape is accurately sinusoidal, the orbits are circular with $\varepsilon = 0$. The distances r_1 and r_2 from the center of mass are constant for circular orbits. The center of mass is the center of the orbits of both stars and of their relative motion.

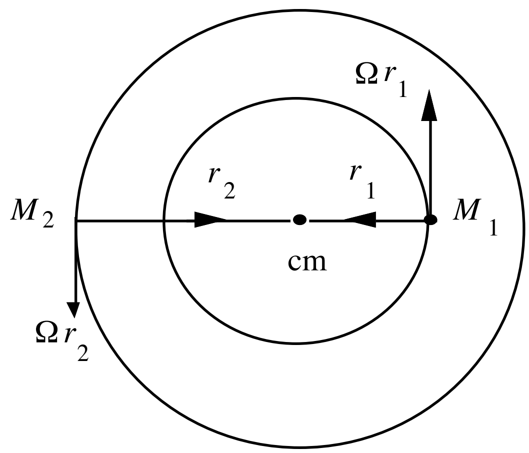


Fig. 4-11

$$r_1 = \frac{M_2}{M_1 + M_2} r \qquad r_2 = \frac{M_1}{M_1 + M_2} r \quad .$$

The distance of M_1 from M_2 is $r_1 + r_2$.

The period and the separation are related by Kepler's Third Law

$$\tau^2 = \frac{4\pi^2 r^3}{GM} = \frac{4\pi^2 (r_1 + r_2)^3}{G(M_1 + M_2)}$$

and the speeds of the stars are

$$\mathrm{v}_1 = \Omega r_1 \, , \quad \mathrm{v}_2 = \Omega r_2$$

where Ω is the angular velocity

$$\Omega = \frac{2\pi}{\tau} .$$

So $r = r_1 + r_2 = (\mathrm{v}_1 + \mathrm{v}_2)/\Omega$

$$\frac{M_1}{M_2} = \frac{r_2}{r_1} = \frac{\mathrm{v}_2}{\mathrm{v}_1} \text{ , independent of } i$$

$$\left(M_1 + M_2\right) = \frac{\Omega^2 r^3}{G} .$$

The peak velocities equal v_1 and v_2 only if the orbital plane is parallel to the line of sight. If i is the inclination angle between the line of sight and the normal to the orbital plane = the angle between the plane of the sky (defined as perpendicular to the line of sight) and the plane of the orbit.

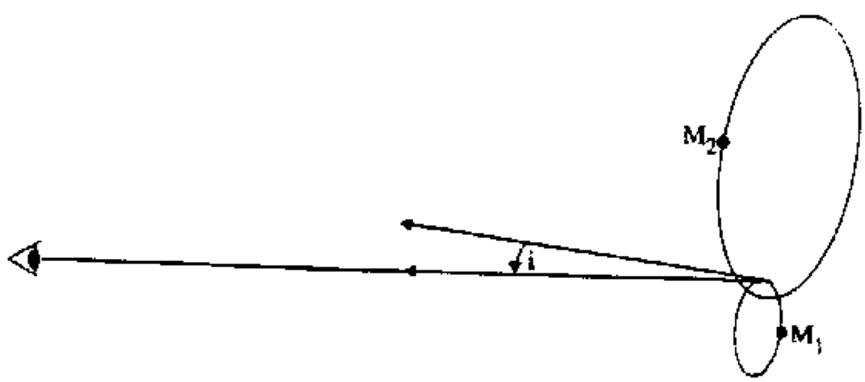


Fig. 4-12

$$K_1 = \mathrm{v}_1 \sin i \;\; = \Omega r_1 \sin i = \frac{2\pi}{\tau} r_1 \sin i$$

$$K_2 = \mathrm{v}_2 \sin i \;\; = \;\; \Omega r_2 \sin i = \frac{2\pi}{\tau} \; r_2 \sin i$$

If $i = 0$, we observe zero velocities (no information). $i = 90°$ is edge on

$K_1 = \mathrm{v}_1$, $K_2 = \mathrm{v}_2$. The mass ratio is in any case independent of i.

$$\frac{M_1}{M_2} = \frac{K_2}{K_1} = \frac{\mathrm{V}_2}{\mathrm{V}_1} \; .$$

The separation $r = r_1 + r_2 = \dfrac{\tau\left(K_1 + K_2\right)}{2\pi \sin i}$

The total mass from

$$M = \frac{\Omega^2 r^3}{G} = \frac{4\pi^2}{\tau^2} \; \frac{r^3}{G}$$

$$M = M_1 + M_2 = \frac{\tau}{2\pi G} \; \frac{\left(K_1 + K_2\right)^3}{\sin^3 \; i} \; .$$

Hence we may write

$$M_1 \sin^3 i = \frac{\tau}{2\pi G} \left(K_1 + K_2 \right)^2 K_2$$

$$M_2 \sin^3 i = \frac{\tau}{2\pi G} \left(K_1 + K_2 \right)^2 K_1 .$$

In general we do not know i.

For eclipsing binaries, each star successively eclipses the other. To see them, i must be near 90°, assuming that the stellar radii are much less than the stellar separation. Masses are insensitive to i for i near 90° since $\sin i \sim 1$.

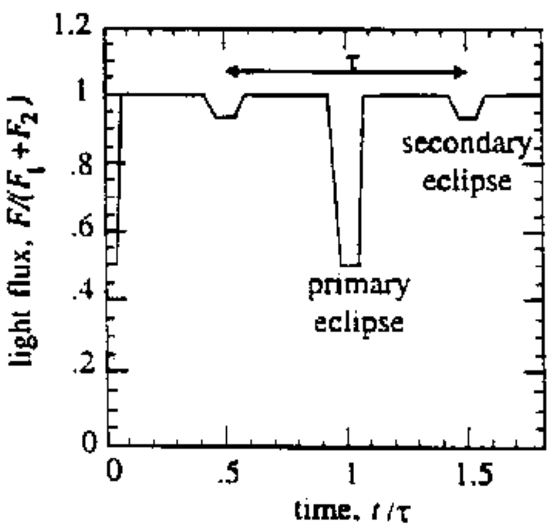


Fig. 4-13

4.3.1 Supernovae in binary systems

Supernovae are exploding stars. Before explosion many occur as binary systems and are caused by mass flow from a companion star. What happens to the binary system when the explosion occurs and the mass of one star is reduced, possibly to zero?

Before, there are two stars in circular orbit

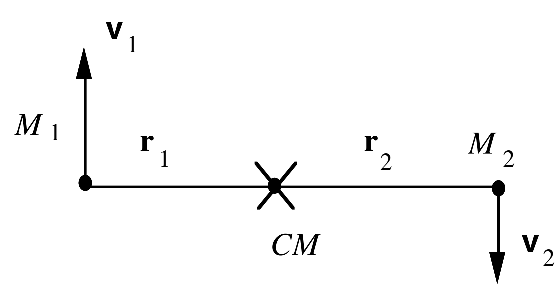


Fig. 4-14

Assume center of mass is at rest, take it as origin

$$M_1 \mathbf{r}_1 + M_2 \mathbf{r}_2 = 0$$
$$M_1 \mathbf{v}_1 + M_2 \mathbf{v}_2 = 0$$

$M_1 > M_2$ explodes, leaving new mass $M_1' = M_1 - \Delta M$. Remaining binary is not at rest. In a spherical explosion, linear momentum carried away is zero. If $\mathbf{v}_c$ is in the new CM velocity, momentum conservation yields

$$M_1' \mathbf{v}_1 + M_2 \mathbf{v}_2 = (M_1' + M_2)\mathbf{v}_c$$

($\mathbf{v}_1$ and $\mathbf{v}_2$ are the same immediately before and after the explosion.) Using $\mathbf{v}_1 = M_2 \mathbf{v}_2 / M_1$ and $M_1' = M_1 - \Delta M$,

$$(M_1 - \Delta M)\left(\frac{-\mathrm{M}_2}{\mathrm{M}_1}\ \mathbf{v}_2\right) + M_2 \mathbf{v}_2$$

$$= (M_1 - \Delta M + M_2)\mathbf{v}_e$$

giving

$$\mathbf{v}_c = \frac{\Delta M\ M_2}{M_1\left(M_1 + M_2 - \Delta M\right)}\ \mathbf{v}_2\ .$$

If $\Delta M = M_1$, $\mathbf{v}_c = \mathbf{v}_2$ (as it must).

Typical values are $M_1 = 10\ M_\odot$, $M_2 = 5\ M_\odot$, $\Delta M = 8.5\ M_\odot$. Then $M_1' = 1.5\ M_\odot$ (appropriate for a neutron star).

$$\mathrm{v}_c = \frac{8.5 \text{ x } 5}{10 \text{ x } 5.5}\mathrm{v}_2\ .$$

For close binaries, $\mathbf{v}_2$ may be several hundred km s^{-1} so system really moves. To determine whether or not the binary remains bound, calculate the binding energy, that is, the internal energy without the center of mass energy.

The total *internal* energy of the system immediately after the explosion is

$$E' = \underbrace{\left(\frac{1}{2} M'_1 \mathrm{v}_1^2 + \frac{1}{2} M_2 \mathrm{v}_2^2\right)}_{\text{Total kinetic energy}} - \underbrace{\frac{1}{2}\left(M'_1 + M_2\right)\mathrm{v}_c^2}_{\text{Energy of CM motion}} - \underbrace{\frac{GM'_1 M_2}{r}}_{\text{Gravitational Energy}}$$

Now $\dfrac{G(M_1 + M_2)}{r} = \left(\mathrm{v}_1 - \mathrm{v}_2\right)^2$ for circular orbits.

We obtain, writing everything in terms of v_2

$$E' = \frac{1}{2}\left(M_1 - \Delta M\right)\left(\frac{M_2 \mathrm{v}_2}{M_1}\right)^2 + \frac{1}{2} M_2 \mathrm{v}_2{}^2$$

$$-\frac{1}{2}\left(M_1 + M_2 - \Delta M\right)\left\{\frac{\Delta M\, M_2 \mathrm{v}_2}{M_1\left(M_1 + M_2 - \Delta M\right)}\right\}^2$$

$$-\frac{\left(M_1 - \Delta M\right) M_2}{M_1 + M_2}\left[\left\{1 + \frac{M_2}{M_1}\right\}\mathrm{v}_2\right]^2$$

which (believe it or not!) simplifies

$$E' = \frac{1}{2}\left(M_2 v_2{}^2\right)\frac{\left(M_1 - \Delta M\right)\left(M_1 + M_2\right)}{M_1^2\left(M_1 + M_2 - \Delta M\right)}\left(M_1 + M_2 - 2\Delta M\right) .$$

All terms are positive except the last factor.

Thus for E to be positive (no binding) , mass ejected $\Delta M > 1/2\ (M_1 + M_2)$.

In the numerical example on p 4.31.

$$8.5 > 1/2\ (10 + 5)$$

and the neutron star departs at high velocity.

Pulsars (rotating neutron stars) often have high velocity as they leave the galactic plane.

The result can be obtained more readily using the *CM* system in which the total energy is

$$E = 1/2\ M\, v^2_{cm} + \mu\, (1/2\ v^2 - GM/r)$$

where v is the relative velocity. Before the explosion for the initial circular orbit

$$v^2 = \frac{GM}{r}\ ,\ \frac{1}{2}v^2 = \frac{GM}{2r}$$

and

$$\frac{E}{\mu} = \frac{1}{2}\mathrm{v}^2 - \frac{GM}{r} = -\frac{1}{2}\frac{GM}{2r} .$$

After explosion, v is unchanged— μ, M and E change as M changes to M' . Internal energy becomes

So

$$\frac{E}{\mu} = \left(\frac{1}{2}\mathrm{v}^2 - \frac{GM'}{r}\right)$$

$$= \left(\frac{1}{2}\mathrm{v}^2 - \frac{GM}{r}\right) + \left(\frac{GM}{r} - \frac{GM'}{r}\right)$$

$$= -\frac{1}{2}\frac{GM}{2r} + \left(\frac{GM}{r} - \frac{GM'}{r}\right)$$

$$= \frac{GM}{2r} - \frac{GM'}{r} .$$

So internal energy > 0 if $M' < M/2$ and ejected mass $\Delta M > M/2$.

4.3.2 Tides

When two bodies are in orbit around each other, the otherwise spherically symmetric gravitational field is distorted by the gravitational attraction of the other body.

4.3.3 Weak tides

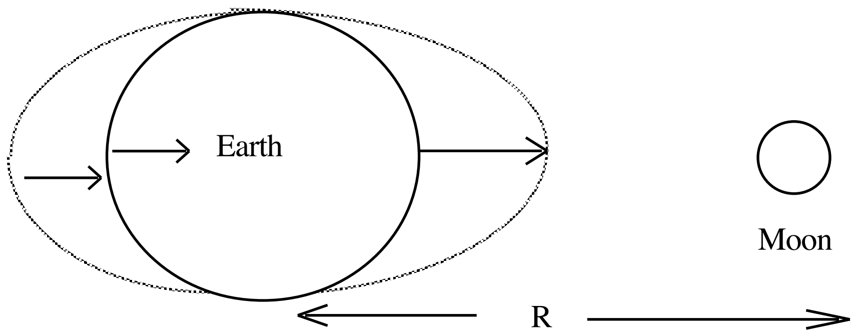


Fig. 4-15

For the Earth-Moon system, the Moon pulls the near surface most strongly, the center of the Earth less strongly and the far surface least strongly. The differential force gives rise to ocean tides.

The ocean surface adjusts to become an equipotential. The potential is formed by the gravitational attraction and by the centrifugal force that arises because the Earth-Moon system is orbiting about the center of mass.

Assume masses are concentrated at the centers of the Earth and Moon. The gravitational potential at a point **r** is

$$V(\mathbf{r}) = \frac{-GM_1}{|\mathbf{r} - \mathbf{r}_1|} - \frac{GM_2}{|\mathbf{r} - \mathbf{r}_2|}$$

to which must be added the centrifugal potential arising because of the rotating frame. It is

$$-\frac{1}{2}r^2\dot{\theta}^2 = -\omega^2 r^2/2$$

where ω is the angular velocity

$$\omega^2 = \frac{G(M_1 + M_2)}{R^3}$$

R being the Earth-Moon distance.

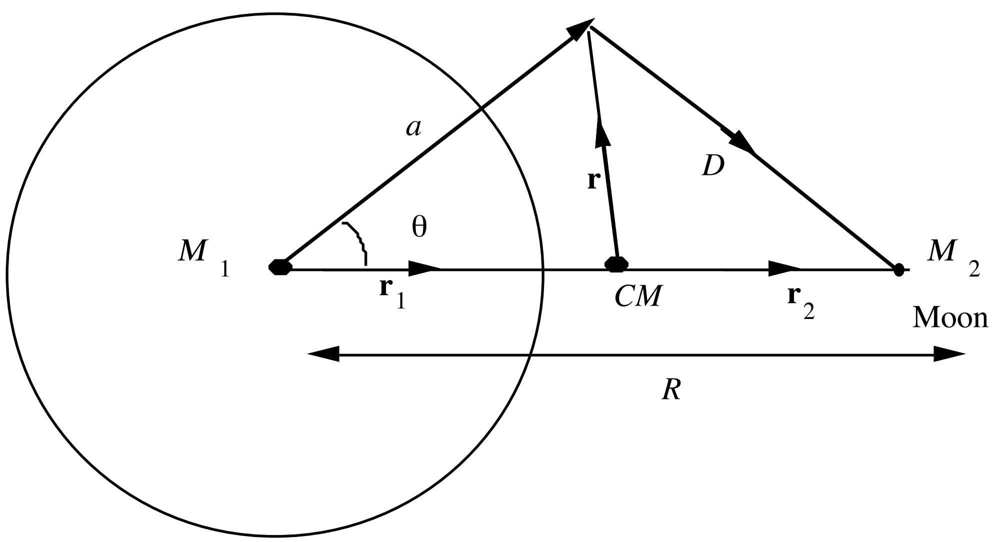


Fig. 4-16

Moon, Earth are small compared to Earth-Moon distance R so we write

$$\frac{-GM_2}{|\boldsymbol{r}-\boldsymbol{r}_2|} = \frac{-GM_2}{D} = \frac{-GM_2}{\left(R^2 + a^2 - 2aR\cos\theta\right)^{1/2}}$$

$$= -\frac{GM_2}{R}\left(1 + \frac{a^2}{R^2} - \frac{2a}{R}\cos\theta\right)^{-1/2} .$$

$$= -\frac{GM_2}{R}\left(1 - \frac{1}{2}\frac{a^2}{R^2} + \frac{a}{R}\cos\theta + \frac{3a^2}{2R^2}\cos\theta^2 + O\left(\frac{a}{R}\right)^3\right)$$

$$= -\frac{GM_2}{R}\left\{1 + \frac{a}{R}P_1\left(\cos\theta\right) + \frac{a^2}{R^2}P_2\left(\cos\theta\right) + \dots\right\}.$$

(Alternatively use expansion

$$\frac{1}{|\boldsymbol{R} - \boldsymbol{a}|} = \frac{1}{R}\sum_n \left(\frac{a}{R}\right)^n P_n(\cos\theta) \qquad R > a$$

where $P_n(\cos\theta)$ are Legendre polynomials.) The term in the potential $\frac{a}{R}\cos\theta$ is linear in z, where z is the direction from M_1 to M_2 so its gradient describes a constant force—GM_2/R^2 which must be canceled by the centrifugal potential.. The centrifugal potential can be written, using $\mathbf{r}_1 = \frac{M_2}{M_1 + M_2}\mathbf{R}$,

$$-\frac{1}{2}\omega^2 r^2 = \frac{1}{2}\omega^2 |\mathbf{r}_1 - \mathbf{a}|^2$$

$$= \frac{1}{2}\omega^2\left[\left(\frac{M_2}{M_1 + M_2}\right)R^2 + a^2 - \frac{2M_2}{M_1 + M_2}R\,a\cos\theta\right] .$$

So adding this we have for the total potential $\Phi(\mathbf{r})$

$$\Phi(\boldsymbol{r}) = -\frac{GM_1}{a} - \frac{GM_2}{R}\left[1 + \frac{a}{R}\, cos\,\theta + 1/2\,(3\cos^2\theta - 1)\,\frac{a^2}{R^2}\right]$$

.

$$\frac{1}{2}\;\frac{G\;(M_1+M_2)}{R^3}\left[\left(\frac{M_2}{M_1+M_2}\right)^2 R^2 + a^2 - 2\left(\frac{M_2}{M_1+M_2}\right) R\,a\,\cos\theta\right]$$

The term in $\frac{a}{R}\cos\theta$ is indeed canceled out by the centrifugal force that keeps the body in a circular orbit. The gradient of terms that do not depend on a or θ is zero, so they may be omitted and we have for the local tidal potential

$$\Phi(\boldsymbol{r}) = -\frac{GM_1}{a} - \frac{1}{2}\frac{Ga^2}{R^3}\left(3\,M_2\,\cos^2\theta + M_1\right).$$

Expand a in terms of its height above the mean sea level $a = R_\oplus + h$. Then

$$a^2 = R_\oplus{}^2\left(1 + \frac{2h}{R_\oplus}\right),$$

$$a^{-1} = R_\oplus\left(1 - \frac{h}{R_\oplus}\right)$$

and

$$\Phi(h,\theta) = -\frac{GM_1}{R_\oplus}\left(1 - \frac{h}{R_\oplus}\right) - \frac{1}{2}\frac{GR_\odot^{\ 2}}{R^3}\left(1 + 2\frac{h}{R_\oplus}\right)\left(3M_2\cos^2\theta + M_1\right)$$

$$\sim \frac{GM_1}{R_\oplus^{\ 2}}\left[h - \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_\oplus}{R}\right)^3 R_\oplus\cos^2\theta\right]$$

ignoring constant terms.

$\frac{GM_1}{R_\oplus^{\ 2}} = g =$ acceleration due to gravity at the surface of the Earth.

The surface is an equipotential so

$$\Phi(h,\theta) = \text{constant}$$

$$h = \frac{3}{2}\left(\frac{M_2}{M_1}\right)\left(\frac{R_\oplus}{R}\right)^3 R_\oplus\cos^2\theta$$

$$+ \text{ constant }.$$

The height of the tides is the difference between high and low values of *h*. Since $\cos^2\theta$ varies between 1 and 0, we get for the height of the tides with

$$\frac{M_2}{M_1} = \frac{1}{81} \;, R_{\oplus} = 6000 \text{ km} = \text{radius of Earth}$$

$$R = 380,000 \text{ km} = \text{Earth -- Moon distance}$$

$$h = 54 \text{ cm} .$$

The same calculation with the Sun in place of the Moon yields

$$h = \frac{3}{2} \text{ x } (332000) \text{ x } \left(\frac{6400}{1.5 \text{ x } 10^8} \right)^3 \text{ x } 6400 \text{ km} = 25 \text{ cm}$$

(presumably by chance, they are of the same order). The tidal effects combine vectorially. When the Moon is at conjunction or opposition, the two forces add to cause the high spring tides.

4.3.4 Tidal friction

The continents are pulled through the ocean bulges and the tidal bulge is dragged ahead by the spinning Earth. There is a loss of energy by friction and the spin of the Earth is slowed. The day is getting longer. (There is evidence from growth scales in fossil corals that there were 400 days in a year about 100 million years ago). Angular momentum is conserved so the Moon increases its angular momentum. It can do so because the non-symmetric bulge creates a gravitational torque back on the Moon. Increasing the angular momentum means the Moon must move outward and so the month is getting longer The lowest energy state of the Moon-Earth system is one in which the Earth and Moon present the same face—in which case the tidal distortion will have reached its equilibrium shape that involves no relative motion of any material. The Earth and Moon will be *tidally locked* and there will be no drag. The tidal bulge will point directly at the Moon and the Earth and the Moon will corotate.

Because the Moon is not exactly spherical, partial locking has already occurred, in that the Moon rotates with the Earth so that it shows the same face all the time. The Moon is in synchronous rotation such that the orbital period of the Moon around the Earth equals the rotation or spin period of the Moon. The ultimate equilibrium caused by tidal friction is that in which the spin velocity of the Earth equals the angular velocity of the Moon in orbit around the Earth (or the angular velocity of the Earth about the moon) so that 1 month equals 1 day.

To calculate when that equilibrium will be reached, use the conservation of angular momentum. The angular momentum of the Earth-Moon system is the sum of the angular momentum of the spinning Earth and Moon and the angular momentum of the Moon's orbit around the Earth.

The angular momentum of the Earth may be written $I_1\omega_1$ where I_1 is the moment of inertia and ω_1 is the spin angular velocity. The angular momentum of a sphere of uniform mass density ρ and radius R spinning about an axis ω is given by

$$L = \left| \int_o^R \int_{-1}^{+1} \int_o^{2\pi} \rho \, \mathbf{r} \times \left(\omega \times \mathbf{r} \right) r^2 dr \, d\left(\cos\theta\right) d\phi \right|$$

$$= 2\pi \, \rho\omega \int_o^R \int_{-1}^{+1} r^4 \left(1 - \cos^2 \theta\right) \, d\left(\cos\theta\right)$$

$$= 2\pi \, \rho R^5 \omega = I_1 \, \omega$$

where $I_1 = 2/5 \, MR^2$ and $M = 4/3 \, \pi R^3 \rho$ is the mass.

Then total angular momentum is

$$I_1 \omega_1 + I_2 \omega_2 + \frac{M_1 M_2}{M} \, \omega_2 R_o^{\,2}$$

(note the Moon spin and orbital angular velocities are equal) and eventually is

$$I_1 \omega_f + I_2 \omega_f + \frac{M_1 M_2}{M} \, \omega_f R_f^{\,2}$$

where $\omega_f = 2\pi$/(ultimate day or month), R_f the ultimate Earth-Moon distance.

1 refers to the Earth and 2 to the Moon and R_o = 380,000 km is the present Earth-Moon distance.

Kepler's law gives R_f

$$\frac{R_f}{R} = \left(\frac{\omega_2}{\omega_f} \right)^{2/3}.$$

Now $M_1 \sim 81.3\ M_2$ so $\frac{M_1 M_2}{M} \sim M_2$.

Also $I_2\omega_2$ is small compared to $I_1\omega_1$ and as we can check once we have the answer, both spins are negligible in the final state. Then

$$\frac{2}{5}\ M_1 R_1{}^2 \omega_1 + M_2\,\omega_2 R_o{}^2 = M_2\,\omega_f R_f{}^2$$

$$= M_2\,\omega_2 R_o{}^2 \left(\frac{\omega_2}{\omega_f}\right)^{1/3}$$

yielding

$$\frac{\omega_2}{\omega_f} = \left\{\frac{\frac{2}{5} M_1 R_1{}^2 \omega_1 + M_2 \omega_2 R_o{}^2}{M_2 \omega_2 R_o{}^2}\right\}^3$$

$$= \left\{1 + \frac{2}{5}\left(\frac{M_1}{M_2}\right)\left(\frac{R_1}{R_o}\right)^2\left(\frac{\omega_1}{\omega_2}\right)\right\}^3$$

$$= \left\{1 + \frac{2}{5} \times 81.3 \left(\frac{6400}{380{,}000}\right)^2 \times 28\right\}^3$$

$$= 1.99$$

(380,000 km is the present Earth-Moon distance). The final length of the day and month will be 28 x 1.99 = 54 days.

The current lengthening of the day is about 0.2 days in 10^9 years so it will take more than 10^{10} years to reach equilibrium. (We will have been engulfed by the Sun in its evolution by then).

The same tidal forces bring binary stars into corotation, tidally locked to each other.

4.3.5 Roche stability limits for satellites

Objects can be torn apart by tidal forces. Tidal potential is

$$\phi(r,\theta) = \frac{-GM_1}{r} - \frac{1}{2}\frac{Gr^2}{R^3}\left(3M_2\cos^2\theta + M_1\right) + \text{constant}$$

Suppose M_1 is the mass of a small satellite orbiting a large parent star or planet of mass M_2. $M_1 \ll M_2$.

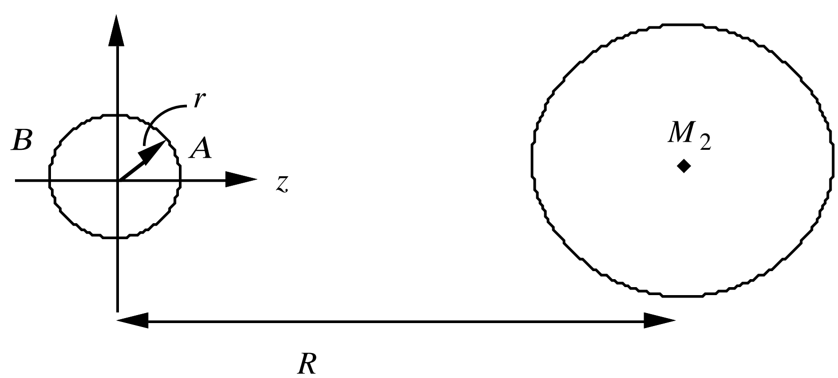


Fig. 4-17

Gravitational acceleration at points A and B is along the z axis and is given by

$$g_z = -\nabla_z \Phi = -\frac{d}{dz}\left[-\frac{GM_1}{|z|} - \frac{1}{2}\frac{Gz^2}{R^3}(3M_2)\right]$$

$$= -\frac{GM_1}{|z|^3}z + z\frac{3GM_2}{R^3}$$

$$= Gz\left(-\frac{M_1}{|z|^3} + \frac{3M_2}{R^3}\right).$$

This is a restoring force if the coefficient of z is negative. If it is positive, the force is away from the center of the satellite and the satellite tends to be torn apart. Putting $|z| = r$ for a satellite of mean density ρ so $M_1 = 4/3\ \pi r^3 \rho$, the condition for *Roche stability* is

$$\rho > \frac{9}{4\pi}\frac{M_2}{R^3}$$

or no satellite of density ρ is stable inside the *Roche radius*.

$$R_{\text{crit}} = \left(\frac{9M_2}{4\pi\rho}\right)^{1/2} .$$

If the parent (planet) and satellite have same density

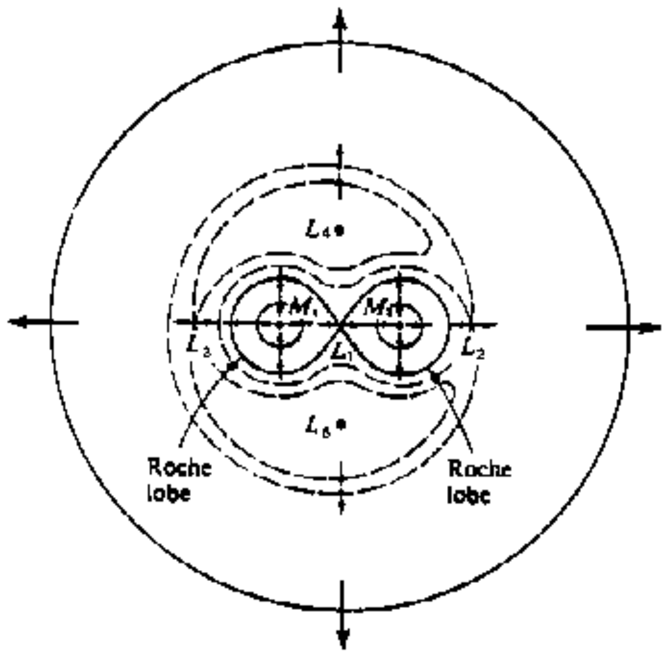
$$R_{\text{crit}} = 3^{1/3} R_2 = 1.44 R_2$$

where R_2 is radius of the parent. (Roche calculated 2.44 R_2 using a model of the self-gravity of the satellite).

This is essentially the physics of the rings of Saturn (and other planets). Material within the Roche limit cannot form bodies such as moons because of the disruptive effect of tidal forces.

4.3.6 Roche lobes

Consider a binary stellar system in a circular orbit. The intersections of the equipotential surfaces with the plane of the orbit are shown in the Figure.



The Roche equipotential surfaces plotted in the equatorial plane for two point mass with a mass ratio equal to 2/3. The short arrows indicate the direction of the effective gravitational field in the frame of reference which corotates with the orbital motion. The effective gravity vanishes at the five Lagrangian points L_1, L_2, L_3, L_4, L_5. The first three, L_1, L_2, L_3, lie along the line joining the two mass points; the last two, L_4, L_5, form equilateral triangles with the two mass points, M_1 and M_2. The sideways "figure 8" which passes through the L_1 point contains the two Roche lobes.

Fig. 4-18

This figure is copied from Shu on p. 186. There are five stationary points, called Lagrangian points, where the force vanishes. Close to each star, the equipotentials are dominated by the gravitational attraction and the equipotentials are circles centered at the stars (taken to be point sources). Far from the stars, the equipotentials are dominated by the outwardly directed centrifugal force. There the equipotentials intersect the equatorial plane in circles enclosing both stars. The two kinds of equipotentials are separated by *Roche lobes* around each star indicated by the figure of eight. The Roche lobes intersect at a saddle point.

Roche lobes can be used to further classify close binaries. If both stars are smaller than their Roche lobes, the system is a *detached* binary.If one fills its Roche lobe, the system is a *semi-detached* binary and matter will flow through the contact point. If both stars fill their Roche lobes they are *contact binaries* and they have a common envelope.

The Roche lobe is the maximum possible size of the star. If a star becomes larger than its Roche lobe, it overflows and dumps mass through the saddle-point on to the companion star.

A common scenario is the case where M_1 is initially much larger than M_2 (possibly also losing mass to infinity in a stellar wind). Mass flows from M_1 to M_2. Eventually M_1 becomes a white dwarf and cools. M_2 has gained mass and so it evolves faster and overflows back on to M_1. This process manifests itself in an X-ray source. As the white dwarf accumulates mass, it may be forced into a gravitational collapse to a neutron star in a supernova explosion.

4.3.7 Effect of mass transfer on binary orbits

Suppose M_1 is filling its Roche lobe and dumping mass on to M_2. Mass and angular momentum are conserved but not energy. The mass is heated and dissipates energy in radiation.

M_2 is gaining mass so $\dot{M}_2 = -\dot{M}_1 > 0$. The angular momentum for a circular orbit

$$L = \mu R^2 \omega = \mu R^2 \left(\frac{GM}{R^3}\right)^{1/2}$$

$$= \frac{M_1 M_2}{M^{1/2}} R^{1/2} G^{1/2} = \left(M_1 M_2 R^{1/2}\right)\left(G^{1/2} M^{-1/2}\right)$$

$$0 = \frac{dL}{dt} = \left(G^{1/2} M^{-1/2}\right)\left(\dot{M}_1 M_2 R^{1/2} + M_1 \dot{M}_2 R^{1/2}\right)$$

$$+ \left(G^{1/2} M^{-1/2}\right)\frac{1}{2} M_1 M_2 \dot{R} \; R^{-1/2} .$$

Solving for $\dot{R}$ and eliminating $\dot{M}_1$ in favor of $\dot{M}_2$, we obtain

$$\dot{R} = 2R\left(\frac{M_2 - M_1}{M_1 M_2}\right)\dot{M}_2 .$$

If the lighter star M_1 is losing mass $\dot{R} > 0$ and stars draw apart. Often this terminates the mass flow since it puts M_1 deeper into its Roche lobe. Alternatively the mass transfer proceeds slowly on the stellar evolutionary time scale that it takes M_1 to fill its increasingly large Roche lobe.

If the heavier star is losing mass $\dot{R}$ is negative. The stars get nearer which increases mass flow leading to a catastrophic instability. In practice friction leads to a merger of the two stars.

4.4 The Virial Theorem

Here I prove a useful theorem, the *virial theorem*.

The kinetic energy of N interacting particles of masses m_i and velocities v_i is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i \mathsf{v}_i^2$$

The gravitational potential energy is

$$\mathsf{V} = -\sum_j \sum_j \frac{G m_i m_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

Newton's law

$$m_i \dot{\mathsf{v}}_i = -\nabla_i \mathsf{V}$$

Introduce

$$I = \frac{1}{2} \sum_{i=1}^{N} m_i r_i^2$$

(similar to the moment of inertia but about a point).

Differentiate *I* with respect to time twice

$$\dot{I} = \sum_{i=1}^{N} m_i \mathsf{v}_i \cdot r_i$$

$$\ddot{I} = \sum_{i=1}^{N} m_i \dot{\mathsf{v}}_i \cdot r_i + \sum_{i=1}^{N} m_i \dot{\mathsf{v}}_i \cdot \dot{\mathsf{v}}_i$$

$$= -\sum_i r_i \cdot \nabla_i \mathsf{V} + 2\mathrm{T} .$$

This is the time-dependent virial theorem. Suppose we scale all $\mathbf{r}_i$ by λ. Then

$$\frac{d}{d\lambda} \mathsf{V}(\lambda \mathbf{r}) = \sum_i \mathbf{r}_i \cdot \nabla_i \mathsf{V} \ .$$

For a gravitational potential

$$\mathsf{V}(\lambda \mathbf{r}) = \frac{1}{\lambda} \mathsf{V}(\mathbf{r})$$

so

$$-\frac{1}{\lambda^2} \mathsf{V}(\mathbf{r}) = \sum_i r_i \nabla_i \mathsf{V} \, .$$

Put $\lambda = 1$. Then

$$\ddot{I} = \mathsf{V} + 2T \, .$$

If a gravitational system is in equilibrium, neither increasing or decreasing in size, it must have the long time average values $\langle \mathsf{V} \rangle$ and $\langle T \rangle$ such that

$$\langle \mathsf{V} + 2T \rangle = 0 \qquad \langle \mathsf{V} \rangle = -2 \langle T \rangle \quad .$$

We can prove this by averaging over a long time Γ

$$\langle V + 2T \rangle = \frac{1}{\Gamma} \int_0^{\Gamma} (V + 2T)dt)$$

$$= \frac{1}{\Gamma} \left[\dot{I}(T) - \dot{I}(0) \right]$$

If Γ is the orbital period, $\dot{I}(T) = \dot{I}(0)$. More generally, if all particles remain bounded with bounded velocities for all time, *I(t)* remains bounded and the right-hand side tends to zero. This relationship $<2T> = -< V>$ applies also to the kinetic and potential energies of many electron atomic systems bound by the Coulomb attraction between the nucleus and the electrons and can be established using quantum mechanics.

(For a harmonic oscillation, $V \sim r^2$,

$$\square V(\lambda \mathbf{r}) \sim \square \lambda^2 V(\mathbf{r})$$

$$\frac{d}{d\lambda} V(\lambda \mathbf{r}) = 2\lambda V(\mathbf{r})$$

$$\lambda = 1$$

$$\ddot{I} = -2V + 2T$$

$$<T> = <V> \quad)$$

4.5 Gravitational Collapse

Imagine cloud mass M, uniform density ρ, radius ρ_o

$$M = \frac{4}{3}\pi r_o^3 \rho$$

held at $r = r_o$ and released. In the absence of other forces, cloud will collapse. Conservation of energy

$$\frac{1}{2}\dot{\mathbf{r}}^2 = \frac{GM}{r} - \frac{GM}{r_o}$$

$$\therefore\ t_{ff} = \int_0^{t_{ff}} dt = -\int_0^{r_o} \left(\frac{dt}{dr}\right) dr$$

$$= \int_0^{r_o} \left[\frac{2GM}{r} - \frac{2GM}{r_o}\right]^{-1/2} dr\ .$$

Substitute $x = r/r_o$

$$t_{ff} = \left(\frac{r_o^3}{2GM}\right)^{1/2} \int_0^1 \left(\frac{x}{1-x}\right)^{1/2} dx\,.$$

Put $x = \sin^2\theta$; integral is $\pi/2$.

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}, \text{ depending only on } \rho\,.$$

Collapse time is independent of initial size.

For Sun, ρ=1.4 gm cm^{-3}

t_{ff} = 1.8 x 10^3 sec = 30 minutes.

Put $x = \sin^2\theta$; integral is $\pi/2$.

$$t_{ff} = \left(\frac{3\pi}{32G\rho}\right)^{1/2}, \text{ depending only on } \rho\,.$$

Collapse time is independent of initial size.

For Sun, ρ=1.4 gm cm^{-3}

tff = 1.8 x 10^3 sec = 30 minutes.