5. Stars and Stellar Structure

5.1 Phenomenology

Essentially all the light we see from the Universe is starlight from the stars or from surrounding material that reflects or is stimulated to emit radiation by the starlight.

Empirically, based on their spectra, the variation of intensity with wavelength, stars are labelled OBAFGKMRNS*, a classification that depends on surface temperature; O stars are the hottest and S stars are the coldest. There are more specific diagnostics in the form of emission lines of different elements in neutral and ionized stages.

5.1.1 Element Abundances

Element abundances provide information about individual stars and the evolution of the Universe. Here is the Periodic Table.

*Only Bungling Astronomers Forget Generally Known Mnenomics

Table 1

Periodic Table

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 0	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 A
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 C	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89 Ac	104	105	106												

58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dv	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

The numbers in the Periodic Table are the atomic numbers defined as the number of protons Z in the nucleus. For neutral atoms, the number of electrons equals the number of protons. Nuclei are made of nucleons. Nucleons are protons or neutrons. They are also called baryons. There are heavier particles that are baryons also but they have mostly decayed into protons. The total number A of protons and neutrons in a nucleus is the atomic mass number and it is written as a superscript placed before the element ${}^{A}X$ (though spoken as XA); e.g. ${}^{12}C$ has six protons and six neutrons and is described as carbon twelve. The isotopic form ${}^{13}C$

has the same charge as ^{12}C and so six protons. To make A = 13, it has seven neutrons. For lighter elements from He (helium) to S(sulfur)

$$A \sim 2Z$$

For heavier elements, A tends to exceed $2Z$ and there are more neutrons than protons. For example, ^{56} Fe(iron) has Z=26.

The chemistry is determined by the charge Z and is (almost) independent of A. The Periodic Table classifies the elements into groups. The columns show elements with similar chemical behavior. Thus the alkali metals Li, Na, K, Rb, Cs, Fr and the inert gases He, Ne, A, Kr, Xe, Ra.

Observations of spectra show that stars can be divided into two major classes, called Population I (Pop One) and Population II (Pop Two). Pop I stars have chemical abundances relative to hydrogen similar to the Sun and are young stars, created by ongoing star-formation within the *Galaxy*—(the Galaxy is our galaxy, the Milky Way) and other galaxies. Population II stars are stars with much lower relative abundances of heavier elements—to astronomers "heavy" means beyond helium in the Periodic Table. Astronomers also often call all heavy elements "metals" so the literature must be read with care). The Pop II stars are old stars formed when the heavy element abundances were low. They may be fossils of the initial epoch of star formation after the Big Bang.

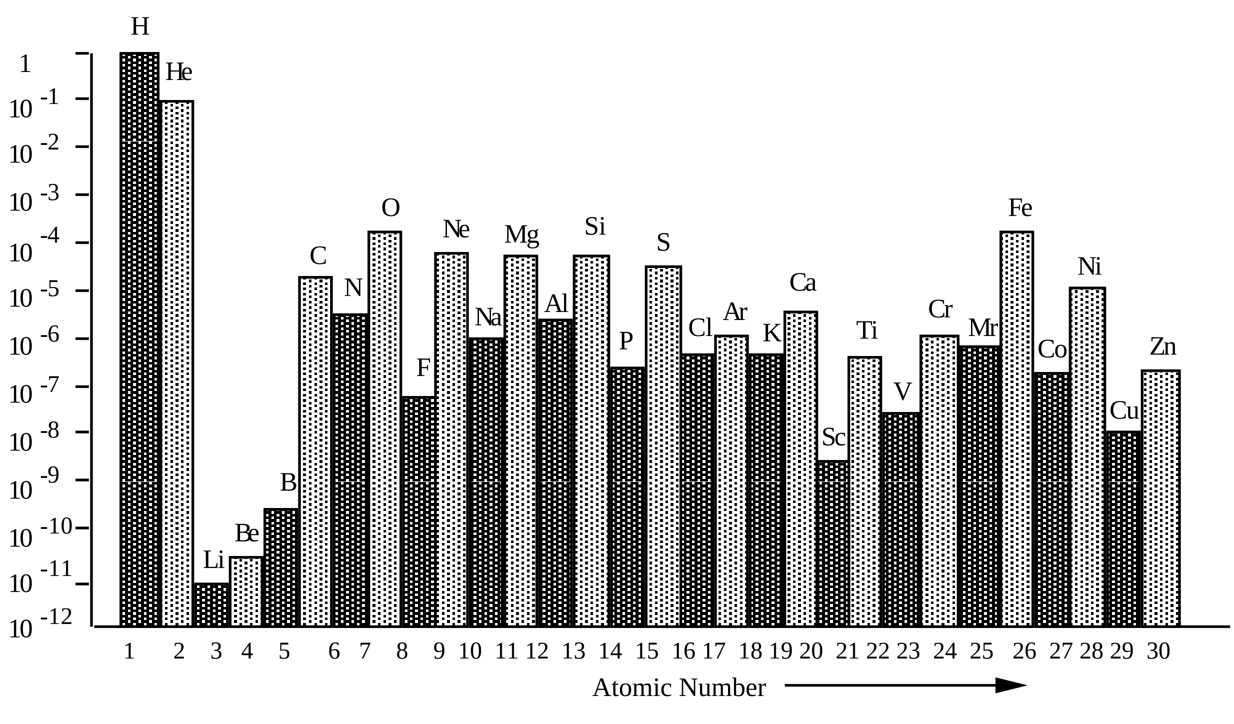
The element abundances of Pop I stars can be obtained from observing the nearest example, the Sun, and also, except that H and He have escaped, from measurements of the composition of the Earth itself.

Table 5-2

Pop I Abundances

	Atomic Number	Mass (main isotope)	Relative Number	Mass Fraction
H	1	1	1	0.77
He	2	4	7×10^{-2}	0.21
C	6	12	4×10^{-4}	4×10^{-3}
N	7	14	9×10^{-5}	1×10^{-3}
O	8	16	7×10^{-4}	9×10^{-3}
Ne	10	20	1×10^{-4}	1×10^{-3}
Mg	12	24	4×10^{-5}	8×10^{-4}
Si	14	28	4×10^{-5}	8×10^{-4}
Fe	26	56	3×10^{-5}	1×10^{-3}

"metals" total 0.02.



Here is a list of Pop I abundances and a diagram of solar abundances on a logarithmic scale. (Abundances are often presented relative to a hydrogen abundance of 10^{12} on a logarithmic scale—e.g. the abundance of carbon would be given as 8.6). Thus

$$\log \frac{n(\mathrm{C})}{n(\mathrm{H})} = 8.6 - 12 = -3.4$$

$$\frac{n(\mathrm{C})}{n(\mathrm{H})} = 10^{-3.4} = 4 \times 10^{-4} .$$

The helium was produced by nucleosynthesis from the protons and neutrons in the Big Bang (and the amount of helium produced then depended on the number of types of light neutrinos). The abundances of Li, Be and B produced in the Big Bang were very small ($< 10^{-10}$) and they have subsequently been partly destroyed in stars by nuclear processes. All heavier elements—carbon and beyond—are made only in stars. The combined mass fraction of heavy elements is usually denoted Z (not to be confused with the nuclear charge). Even elements with equal numbers of protons and neutrons tend to be more abundant than odd elements—because a major building block in nucleosynthesis is the ^{4}He nucleus—α-particles—consisting of two protons and two neutrons.

Pop I stars have $Z \sim 0.02$ and are made of material with a heavy element abundance that has been enriched by processing in earlier generations of stars (which have distributed their material back into the interstellar medium through various ejection events, including supernovae).

In contrast Pop II stars which have Z as low or lower than 0.002 may comprise the survivors of the original generation of stars. (There may have been a still earlier generation, referred to as Pop III).

Our galaxy has the shape of a thin disk (thickness ~ 200 pc, radius ~ 8 kpc) with a bulge and a halo of stars.

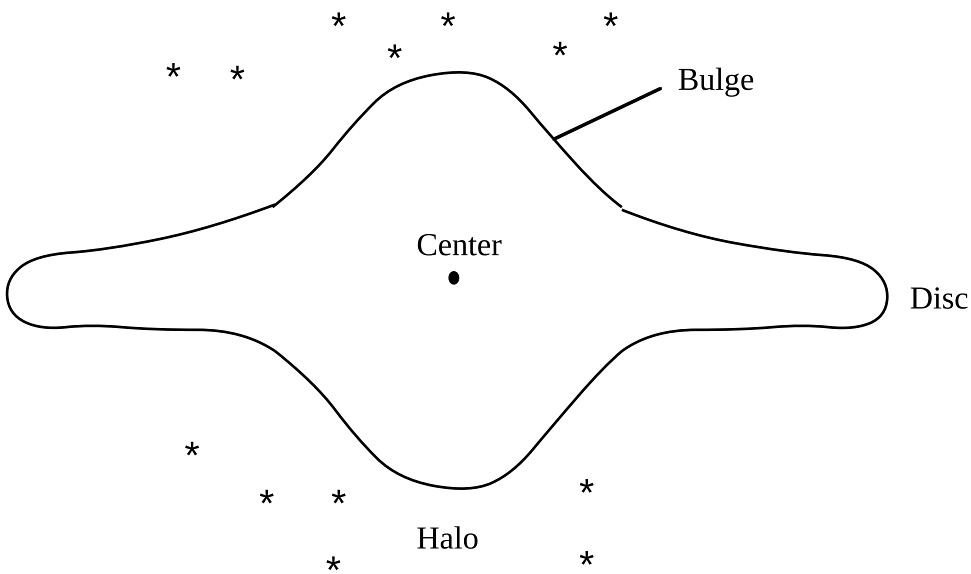


Fig. 5.1

Pop I stars are located mostly in the disk and Pop II stars in the bulge and halo.

5.1.2 Nuclear reactions

Nuclear reactions involve protons, neutrons, positrons, neutrinos and photons (γ-rays). Electrons, muons and neutrinos are leptons (light particles) with lepton number +1. Positrons, anti-muons and antineutrinos have lepton number -1. Electrons and anti-muons are negatively charged, positrons and muons are positively charged. Neutrinos and antineutrinos are neutral. In nuclear reactions, electric charge, atomic mass number and lepton mass number are conserved.

The stars are powered by nuclear reactions that transmute (or burn) lighter elements into heavier elements. The energy we receive as starlight originates in nuclear reactions.

Main sequence stars are powered by the conversion of four hydrogen nuclei-protons—into one helium nucleus-alpha-particle. The process releases energy because the ^{4}He nuclei weighs less than four protons. In atomic mass units (AMU; by definition ^{12}C has a mass of 12 AMU), M_{H} = 1.0078 and M_{He} = 4.0026. Thus $4M_{\mathrm{H}} - M_{\mathrm{He}} = 0.0286$ AMU and 0.71% of the mass of each proton is converted to energy ($E=mc^2$). Unit atomic weight (1 AMU) is 1.66×10^{-27} kg, so energy released per helium nucleus (α-particle) formed is

$$0.0286(1.66 \times 10^{-27})\,(9 \times 10^{16}) \text{ Joules}$$

$$= 4.3 \times 10^{-12} \text{ J} = 4.3 \times 10^{-5} \text{ ergs}$$

$$(1\text{J} = 10^{7} \text{ ergs, } 1\text{eV} = 1.6 \times 10^{-12} \text{ ergs,}$$

$$1\text{AMU} \times c^2 = 1.4918 \times 10^{-3} \text{ ergs}$$

$= 931.1$ MeV)

In the Sun, only in the $0.1 M_\odot$ core is the temperature and pressure high enough for the fusion reactions to proceed. So the total thermonuclear energy available is

$$0.0071\ (9 \times 10^{16})\ (0.1 M_\odot)$$
$$= 10^{44}\ \text{J} = 10^{51}\ \text{ergs}\quad .$$

Present solar luminosity is 3.90×10^{33} erg s^{-1}, so the Sun will be sustained for 10 billion years, twice its present age.

The actual reaction sequence is the *p-p* (proton-proton) cycle which dominates nucleosynthesis in lower mass stars ($M < 1.5\ M_\odot$) It is the sequence of two-body reactions

$$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{D} + e^+ + \nu \quad (1.44\ \text{MeV}) \qquad (5.1)$$

$$^2\text{D} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma \quad (5.49\text{MeV}) \qquad (5.2)$$

$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^1\text{H} + {}^1\text{H} \quad (12.9\ \text{MeV})\quad . \qquad (5.3)$$

The γ photon is needed in the second reaction to conserve momentum. The neutrino ν is needed in the first reaction to conserve lepton number. You can check that A is conserved by adding the prefixes on each nucleus. The energies in parentheses are the energies released in the reaction. The energy gained by the neutrino escapes. The rest of the energies are converted into thermal energy of the star.

Further reactions occur: ^{7}Be is made from

$$^{3}\text{He} + {}^{4}\text{He} \rightarrow {}^{7}\text{Be} + \gamma \; . \tag{5.4}$$

The sequence

$$^{1}\text{H} + {}^{7}\text{Be} \rightarrow {}^{8}\text{B} + \gamma \tag{5.5}$$

$$^{8}\text{B} \rightarrow {}^{8}\text{Be} + e^{+} + \nu \tag{5.6}$$

yields a neutrino escaping with energy 7.2 MeV. The measured neutrino flux is ~ half or less than (reliable) solar models predict—this is the solar neutrino problem.

Higher mass stars (> 1.5 $M_{\odot}$) burn via the so-called CNO cycle in which hydrogen is again converted to helium but with carbon acting as a catalyst. The main reactions are

$$^{12}\text{C} + {}^{1}\text{H} \rightarrow {}^{13}\text{N} + \gamma \tag{5.7}$$

followed by the spontaneous decay of ^{13}N (the stable isotopes of nitrogen are ^{14}N

and ^{15}N)

$$^{13}\text{N} \rightarrow {}^{13}\text{C} + e^{+} + \nu \; . \tag{5.8}$$

Then

$$^{13}C + {}^{1}H \rightarrow {}^{14}N + \gamma \tag{5.9}$$

$$^{14}N + {}^{1}H \rightarrow {}^{15}O + \gamma \tag{5.10}$$

(^{16}O, ^{17}O, ^{18}O are the stable isotopes of O)

$$^{15}O \rightarrow {}^{15}N + e^{+} + \nu \tag{5.11}$$

$$^{15}N + {}^{1}H \rightarrow {}^{12}C + {}^{4}He \ . \tag{5.12}$$

The ^{12}C is recovered and ^{1}H is converted to ^{4}He. Overall

$$\backslash 4 \times {}^{1}H \rightarrow {}^{4}He + 2e^{+} + 2\nu \tag{5.13}$$

plus γ rays.

Thermal fusion reactions make the elements up to Fe. Beyond iron requires an input of energy, say, in supernova explosions.

Because of the Coulomb repulsion, nuclear reaction rates are extremely sensitive to temperature. Above some threshold energy, a small increase in temperature causes a large increase in the reaction rate. The temperature is determined by a balance of the heating and cooling rates. The cooling rate increases exponentially with *T*. Stars are accordingly thermally stable—the central temperatures where the nuclear reactions occur vary across a wide range of stellar masses in the narrow temperature range 1×10^7K - 2×10^7 K. Cooling reactions increase rapidly in efficiency as T increases between 1 and 2×10^7 K.

A good empirical approximation for the interior or central temperature of main sequence stars is

$$T = 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^{1/3} \text{K} \quad . \tag{5.14}$$

Temperature increases with mass. Detailed models of stellar structure and radiation transport are needed to explain this relationship.

5.1.3 Collapse of a massive star - Type II supernova

The hydrogen burns to helium at the hydrogen-helium interface and as the hydrogen is consumed the star contracts and the temperature rises until it reaches the temperature at which He burns into C. The sequence continues through C to O to Si to Fe, producing an onion-like layered structure with iron in the core, surrounded by a mantle of Si, O and C and an envelope of He and H. (Fig. 2) The mean hydrogen-helium burning time is 10^7 years. Oxygen burning takes 200 days and silicon burning takes 2 days.

Iron is not a fuel—there are no exothermic reaction sinvolving iron that release energy so collapse becomes inevitable. The iron nuclei are broken down into their basic constituents of protons and neutrons by photodisintegration processes such as

$$^{56}\mathrm{Fe} + \gamma \rightarrow \ ^{13}4\mathrm{He} + 4n \qquad (5.15)$$

$$^{4}\mathrm{He} + \gamma \rightarrow \ 2p + 2n\,, \qquad (5.16)$$

processes which remove energy fom the core and cool it. The collapse is held up temporarily by electron degeneracy pressure (see §5.3.3) but the protons capture the electrons and a neutron core is formed. At this point the outer layers are suspended above a central core on which they collapse. The star explodes as rebound shock drives the material out into the surrounding medium. Left behind is either a neutron star or a black hole, which depends on the mass. If M is greater than 25 $M_{\odot}$, a black hole is formed, if less, then a neutron star, supported by neutron degeneracy pressure.

During the explosion, explosive nucleosynthesis creates ^{56}Ni which decays in 8 days to ^{56}Co which decays in 112 days to ^{56}Fe, both decays producing γ-rays and positrons. The γ-rays and positrons are the source of the continuing luminosity of the ejecta. For SN 1987a, the luminosity is now driven by positrons emitted in the decay of ^{44}Ti, the γ-rays escaping.

The total energy released in the explosion is of the order of 10^{52} ergs, of which 1% is converted to light and the rest is carried away by neutrinos.

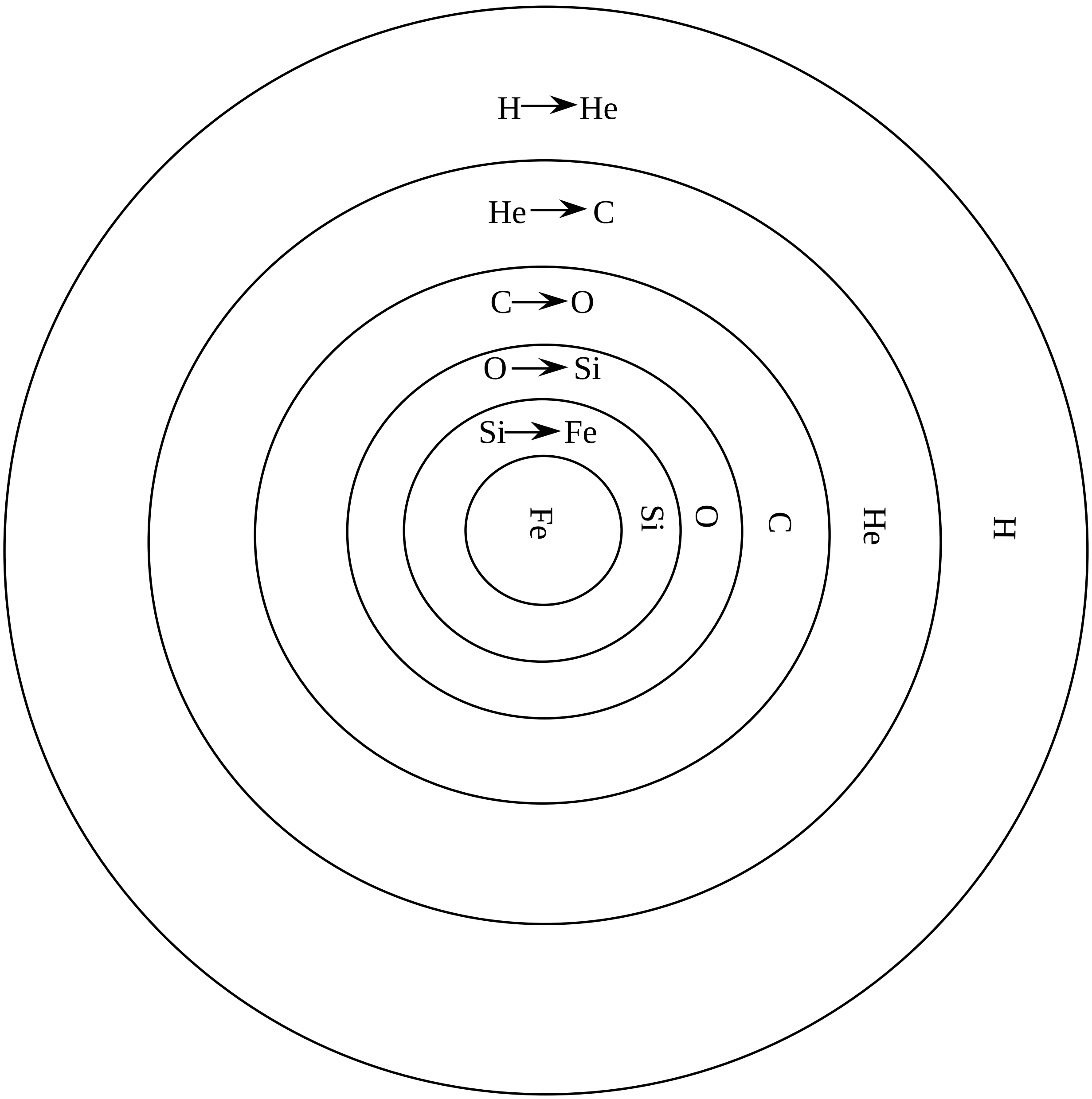
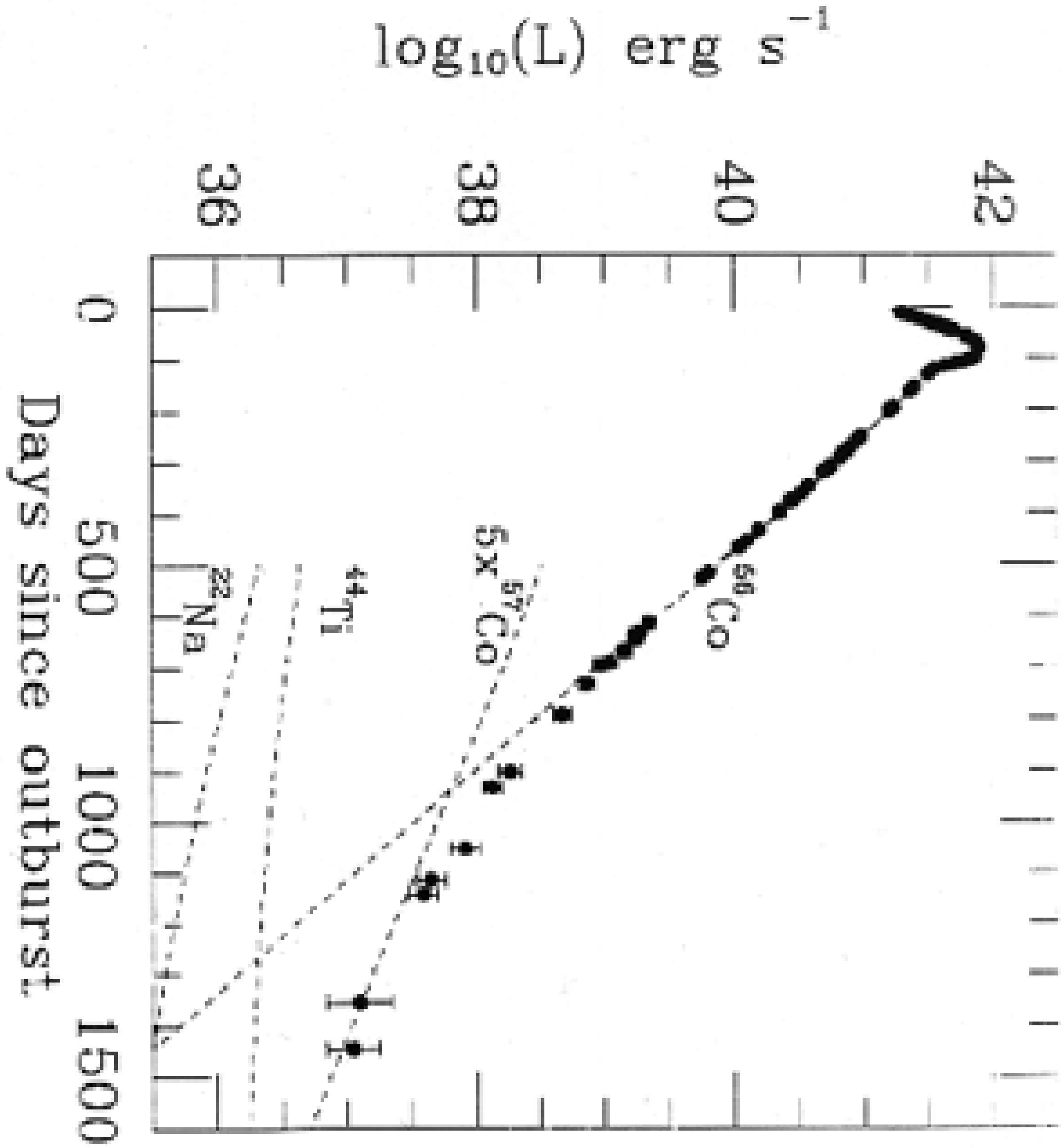


Fig. 5-2



log10(L) erg s-1
42
40
38
36
Days since outburst
0
500
1000
1500
56Co
5x 57Co
44Ti
22Na

Fig. 5-3

5.2 Stellar Structure

5.2.1 Order of magnitude

We can use the virial theorem (p. 4.49) to get an approximate relationship between the mass and radius of a star. The theorem asserts that the total energy is half the gravitational potential energy.

The potential energy is

$$P.E. \;=\; \int\int \frac{G dm_1 dm_2}{r_{12}} \sim \frac{GM^2}{R} \tag{5.17}$$

where R is the characteristic size—the radius—and M is the mass.

Now we will show later that

$$\begin{aligned} K.E. \;&=\; \frac{3}{2} N_{particles}\, kT \\ &\sim \frac{M}{m_p}\, kT\,, \end{aligned} \tag{5.18}$$

m_p is the proton mass.

Equating $P.E.$ and $K.E.$ $GM^2/R = \frac{M}{m_p} kT$ and the T-M relationship (eqn. 5-14), we relate radius to mass.

$$R \sim \frac{GMm_p}{kT} \sim \frac{GM_\odot m_p}{k(15 \times 10^6\ K)} \left(\frac{M}{M_\odot}\right)^{2/3}$$

$$= \frac{\left(6.67 \times 10^{-8}\right)\left(1.99\times 10^{33}\right)\left(1.67 \times 10^{-24}\right)}{\left(1.38 \times 10^{-16}\right)1.5\times 10^{7}} \left(\frac{M}{M_\odot}\right)^{2/3} \text{cm}$$

$$= 1.0 \times 10^{11}\left(\frac{M}{M_\odot}\right)^{2/3} \text{cm} \ . \qquad (5.19)$$

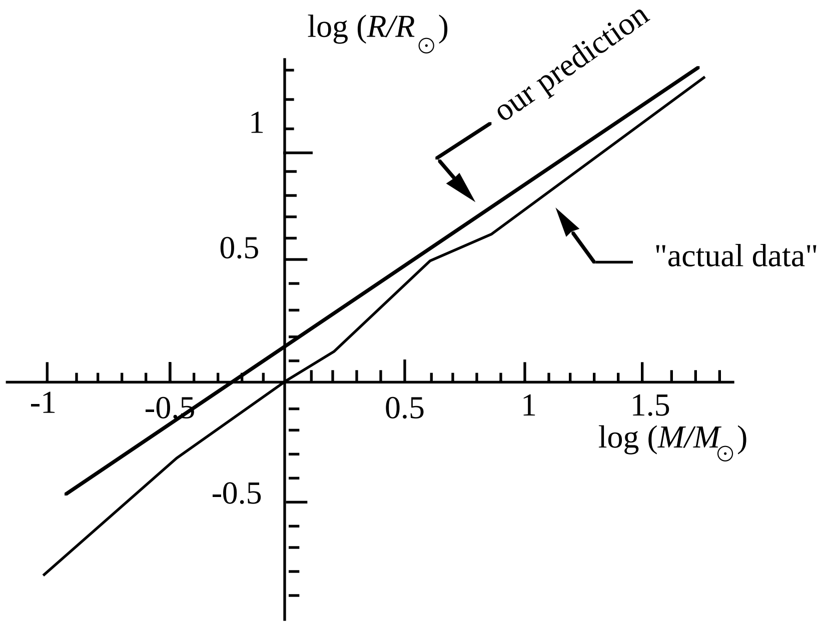
In the case of the Sun, $M = M_\odot$, the actual radius is 7×10^{10} cm.

The Table shows the data for typical main sequence stars

Table 5-3

Physical Properties of Main-Sequence Stars

Log ($M/M_{\odot}$)	Spectral class	Log ($L/L_{\odot}$)	M_{bol}	M_{V}	Log ($R/R_{\odot}$)
-1.0	M6	-2.9	12.1	15.5	-0.9
-0.8	M5	-2.5	10.9	13.9	-0.7
-0.6	M4	-2.0	9.7	12.2	-0.5
-0.4	M2	-1.5	8.4	10.2	-0.3
-0.2	K5	-0.8	6.6	7.5	-0.14
0.0	G2	0.0	4.7	4.8	0.00
0.2	F0	0.8	2.7	2.7	0.10
0.4	A2	1.6	0.7	1.1	0.32
0.6	B8	2.3	-1.1	-2.2	0.49
0.8	B5	3.0	-2.9	-1.1	0.58
1.0	B3	3.7	-4.6	-2.2	0.72
1.2	B0	4.4	-6.3	-3.4	0.86
1.4	08	4.9	-7.6	-4.6	1.00
1.6	05	5.4	-8.9	-5.6	1.15
1.8	04	6.0	-10.2	-6.3	1.3



Mass-Radius Relation for Stars

Fig. 5-4

There is also a mass-luminosity relationship. Very approximately,

$$L \sim M^3 \tag{5-20}$$

for low mass stars but to prove it also requires a consideration of the transport of radiation from the interior to the surface.

5.2.2 Stellar interiors

A star is a self-gravitating gaseous system, its interior so hot the material is ionized to nuclei and electrons constituting a fully ionized *plasma*. The quantities needed to specify a stellar interior as a function of radius from the center r are

- density $\rho(r)$ - decreasing to zero at the surface. In a gas consisting of n_X particles of mass m_X in unit volume, the total number of particles per unit volume—the particle number density— $n = \sum_X n_X$. The mass density $\rho = \sum n_X m_X$ in units of mass per unit volume—g cm^{-3}.
- mass $M(r)$ - interior to r. At $r = 0$, $M(r) = 0$ *and at the surface* $M(r)$=M, *the mass of the star*
- pressure $P(r)$ - Pressure is force per unit area. Weight is the force created by a gravitational field.The gravitational pressure at r is the total weight per unit area of the overlying mass. It is resisted by the pressure generated by the motions of the particles of the gas.
- temperature T(r) - collisions are frequent and we may assume that at r local thermodynamic equilibrium prevails corresponding to the temperature $T(r)$.
- luminosity L - luminosity is the net outward total energy flow at r. It is zero at the source of its generation ($r \sim 0$) and is constant from there outwards to the surface.

- mean particle mass $\bar{m}(r)$ or mean molecular weight - It is the average mass per unit volume per particle. In terms of it, the gas pressure $P(r) = (\rho/\bar{m})kT$ $= nkT$ where n is the particle density. $\left(\bar{m} = \sum_x n_x m_x / n = \rho / n\right)$.

- the opacity $\kappa_\nu(r)$, also called the mass absorption coefficient. It has units of cm^2 per gram. It is the cross sectional area for absorbing or scattering of photons of frequency ν by a gram of material. Travelling a path length ds, the specific intensity I_ν is decreased according to $dI_\nu/ds = -\kappa_\nu \rho I_\nu$.
- nuclear energy generation rate $\varepsilon(r)$ in ergs $cm^{-3} s^{-1}$.

5.2.3 Equations of stellar structure

Assume Hydrostatic Equilibrium.

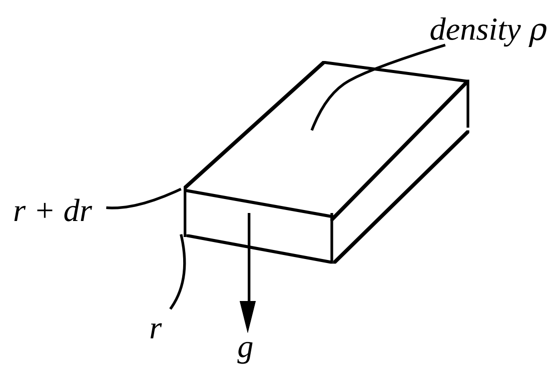


Fig. 5-5

Consider an element of gas between r and $r + dr$ from the center of a star. Pressure $P(r)$ is larger than $P(r + dr)$ by the weight per unit area of the material between r and $r + dr$ in the local gravitational acceleration $g(r)$. If the area of the element is dA, mass of the element is $\rho dAdr$ and weight is $g(r)\rho(\mathrm{r})dAdr$ = $\frac{GM(r)}{r^2}\rho(r)\, dA\ \ dr$ where ρ is the mass density. Thus, the weight of the element per unit area is

$$\frac{G\rho(r)M(r)}{r^2}\ dr. \tag{5.21}$$

The derivative of the pressure is

$$\frac{dP}{dr} = \frac{P(r+dr)-P(r)}{dr} = -\ \frac{G\rho(r)M(r)}{r^2} \tag{5.22}$$

(insert minus sign because pressure increases as r gets smaller). This is the equation of *hydrostatic equilibrium*.

It can also be written

$$\frac{dP}{dr} = -\ \rho g \tag{5.23}$$

where $g = GM(r)/r^2$ is the local acceleration of gravity at r.

We can get an estimate of the central pressure if we assume ρ is constant out to a radius R (in practice it decreases outwards). If ρ is constant

$$\frac{dP}{dr} = -\frac{G}{r^2}\frac{4}{3}\pi r^3 \rho \times \rho$$
$$= -\frac{4}{3}\pi G \rho^2 r \qquad (5.24)$$

Solution must satisfy the boundary condition $P(R) = 0$.

$$\therefore\ P(r) = \frac{2}{3}\pi G \rho^2 \left(R^2 - r^2\right) \qquad (5.25)$$

Central pressure is $P_c = P(r{=}0)$:

$$P_c \sim \frac{2}{3}\pi G \rho^2 R^2 = \frac{3GM^2}{8\pi R^4} \qquad (5.26)$$

and

$$P(r) = P_c(1 - r^2/R^2)\ . \qquad (5.27)$$

Mass equation

Mass interior to r is

$$M(r) = \int_0^r \rho(r')4\pi r'^2 dr' \qquad (5.28)$$

or equivalently

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)\ . \qquad (5.29)$$

To complete the equations we need P as a function of ρ.

5.3 Equation of State

Consider a gas with density ρ and temperature T. The internal pressure of the gas depends on the distribution of energy between kinetic energy and internal energy, which in turn depends upon ρ and T. So

$$P = P(\rho, T) \ . \tag{5-30}$$

This equation which expresses the pressure in terms of ρ and T is the *Equation of State*. It depends on the composition.

For a perfect gas

$$P = nKT = \frac{\rho kT}{\overline{\overline{m}}} \tag{5-31}$$

where $\boldsymbol{m} = \rho / n$ is the mean molecular weight. If the gas is embedded in a black body radiation field with temperature T_{rad}

$$P = \frac{\rho kT}{\overline{\overline{m}}} + \frac{1}{3} a \, T^4_{rad} \tag{5-32}$$

The temperature is determined by an array of heating and cooling processes. For a gas of uniform composition in which the gas pressure substantially exceeds the radiation pressure, T is determined by ρ. So P which is a function of ρ and T can be expressed as a function of P alone (or T alone). In such cases, the pressure is often written as

$$P = K\rho^{1+1/n} \tag{5-33}$$

where K and n are constants which contains all the physics of the heating and cooling processes.

This equation in which the relationship between P and ρ is a pure power law is called a polytropic equation of state. The parameter n which need not be an integer is called the polytropic index. (The notation arises from a combination of the perfect gas law $P \; \alpha \; \rho \, T$ and an assumed relationsihp $T \, \alpha \, \rho^{1/n}$.) It happens that main sequence stars are modeled reasonably well by n = 3 polytropes.

5.4 The Perfect (Ideal) Gas Law

The law is determined experimentally to be

$$PV=NkT \tag{5-34}$$

where V is the gas volume and N is the total number of particles in V. The number density $n = n/V$ (n is not to be confused with the polytropic index).

In terms of n,

$$P = n \, K \, T. \tag{5-35}$$

We can, with one assumption, prove it using the same arguments as in §3.1.4 on radiation pressure, except we have particles in place of photons and the energy is 1/2 $m\,\mathrm{v}^2$. Then

$$P = \frac{2}{3}\,u \tag{5-36}$$

where u is the energy density. Here the energy density depends upon the distribution of velocities of the particles of the gas $n(\mathrm{v})d\,\mathrm{v}$:

$$u = \frac{1}{2}\int_0^\infty \mathbf{p}\cdot\mathbf{v}\;n(\mathrm{v})\,d\mathrm{v} \tag{5-37}$$

and

$$n = \int_0^\infty n(\mathrm{v})\,d\mathrm{v}\ . \tag{5-38}$$

For non-relativistic particles, $p = m\mathrm{v}$

$$P = \frac{2}{3}u = \frac{1}{3}\int_0^\infty m\,\mathrm{v}^2\;n(\mathrm{v})\,d\mathrm{v}\ . \tag{5-39}$$

We now assume we know that $n(\mathrm{v})\,d\,\mathrm{v}$ is the Maxwell distribution function

$$n(\mathrm{v})\,d\mathrm{v} = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m\mathrm{v}^2/2kT}\ 4\pi\,\mathrm{v}^2\,d\mathrm{v}\,. \tag{5-40}$$

Then $\int n(\mathrm{v})d\mathrm{v} = n.$.

Substituting into the integral for P we obtain

$$P = nkT\ . \tag{5-41}$$

The energy density can be given as a mean energy $\overline{E}$ per particle by writing

$$\overline{E} = \frac{1}{n}\int n(\mathrm{v})E\,d\mathrm{v} \tag{5-42}$$

or

$$\overline{\mathrm{v}^2} = \frac{1}{n}\int n(\mathrm{v})\mathrm{v}^2\ d\mathrm{v}\,. \tag{5-43}$$

Evaluating the integral, we obtain

$$\overline{\mathrm{v}^2} = \frac{3kT}{m}\ . \tag{5-44}$$

Thus, mean kinetic energy per particle is

$$\frac{1}{2}m\,\overline{\mathrm{v}^2} = \frac{3kT}{2} \tag{5-45}$$

(each degree of freedom contributes $kT / 2$ to the energy).

If the gas contains particles of different mass, we write $n = \rho/\overline{m}$ where ρ is the mass density and $\overline{m}$ is the average mass often expressed as a *mean molecular weight* μ, in units of the mass of the hydrogen atom

$$\mu = \overline{m} / m_{\mathrm{H}} \quad , \tag{5-46}$$

$m_{\mathrm{H}} = 1.673625 \times 10^{-24}\ g$ is the mass of a hydrogen atom. Then

$$P = \frac{\rho k T}{\mu m_{\mathrm{H}}} \quad . \tag{5-47}$$

The mean molecular weight depends on composition, and on the state of ionization, because free electrons are included in the definition of $\overline{m}$.

For a neutral gas with N_j particles j and masses $A_j = m_j/m_{\mathrm{H}}$ relative to hydrogen

$$\mu_n m_{\mathrm{H}} = \frac{\sum_j N_j A_j m_{\mathrm{H}}}{\sum_j N_j} \tag{5-48}$$

so

S

$$\mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j} . \tag{5-49}$$

For a completely ionized gas

$$\mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j \left(1 + Z_J\right)} . \quad \text{(electron mass may be ignored)} \tag{5-50}$$

where Z_j is the nuclear charge (electron mass is negligible).

Introduce *mass fractions*

$$X = \frac{\rho_H}{\rho} = \frac{\text{total mass of hydrogen}}{\text{total mass}} . \tag{5-51}$$

Similarly Y for helium and Z for elements heavier than He. Thus $X + Y + Z = 1$.

$$Xj = \frac{\rho_j}{\rho} = \frac{\text{total mass of particle } j}{\text{total mass}} = \frac{N_j m_j}{\sum_j N_j m_j} .$$

$$X_H = X, X_{He} = \mathrm{Y}, X_{other} = Z . \tag{5-52}$$

Then write