5. Stars and Stellar Structure

5.1 Phenomenology

Essentially all the light we see from the Universe is starlight from the stars or from surrounding material that reflects or is stimulated to emit radiation by the starlight.

Empirically, based on their spectra, the variation of intensity with wavelength, stars are labelled OBAFGKMRNS, a classification that depends on surface temperature; O stars are the hottest and S stars are the coldest. There are more specific diagnostics in the form of emission lines of different elements in neutral and ionized stages.

5.1.1 Element Abundances

Element abundances provide information about individual stars and the evolution of the Universe. Here is the Periodic Table.

1 H																	2 He
3 Li	4 Be											5 B	6 C	7 N	8 0	9 F	10 Ne
11 Na	12 Mg											13 Al	14 Si	15 P	16 S	17 Cl	18 A
19 K	20 Ca	21 Sc	22 Ti	23 V	24 Cr	25 Mn	26 Fe	27 Co	28 Ni	29 C	30 Zn	31 Ga	32 Ge	33 As	34 Se	35 Br	36 Kr
37 Rb	38 Sr	39 Y	40 Zr	41 Nb	42 Mo	43 Tc	44 Ru	45 Rh	46 Pd	47 Ag	48 Cd	49 In	50 Sn	51 Sb	52 Te	53 I	54 Xe
55 Cs	56 Ba	57 La	72 Hf	73 Ta	74 W	75 Re	76 Os	77 Ir	78 Pt	79 Au	80 Hg	81 Tl	82 Pb	83 Bi	84 Po	85 At	86 Rn
87 Fr	88 Ra	89 Ac	104	105	106												
				58 Ce	59 Pr	60 Nd	61 Pm	62 Sm	63 Eu	64 Gd	65 Tb	66 Dv	67 Ho	68 Er	69 Tm	70 Yb	71 Lu
				90 Th	91 Pa	92 U	93 Np	94 Pu	95 Am	96 Cm	97 Bk	98 Cf	99 Es	100 Fm	101 Md	102 No	103 Lr

The numbers in the Periodic Table are the atomic numbers defined as the number of protons Z in the nucleus. For neutral atoms, the number of electrons equals the number of protons. Nuclei are made of nucleons. Nucleons are protons or neutrons. They are also called baryons. There are heavier particles that are baryons also but they have mostly decayed into protons. The total number A of protons and neutrons in a nucleus is the atomic mass number and it is written as a superscript placed before the element ^{A}X (though spoken as XA); e.g. ^{12}C has six protons and six neutrons and is described as carbon twelve. The isotopic form ^{13}C has the same charge as ^{12}C and so six protons. To make A = 13, it has seven neutrons. For lighter elements from He (helium) to S(sulfur)

$$A \sim 2Z$$

For heavier elements, A tends to exceed $2Z$ and there are more neutrons than protons. For example, ^{56}Fe(iron) has Z=26.

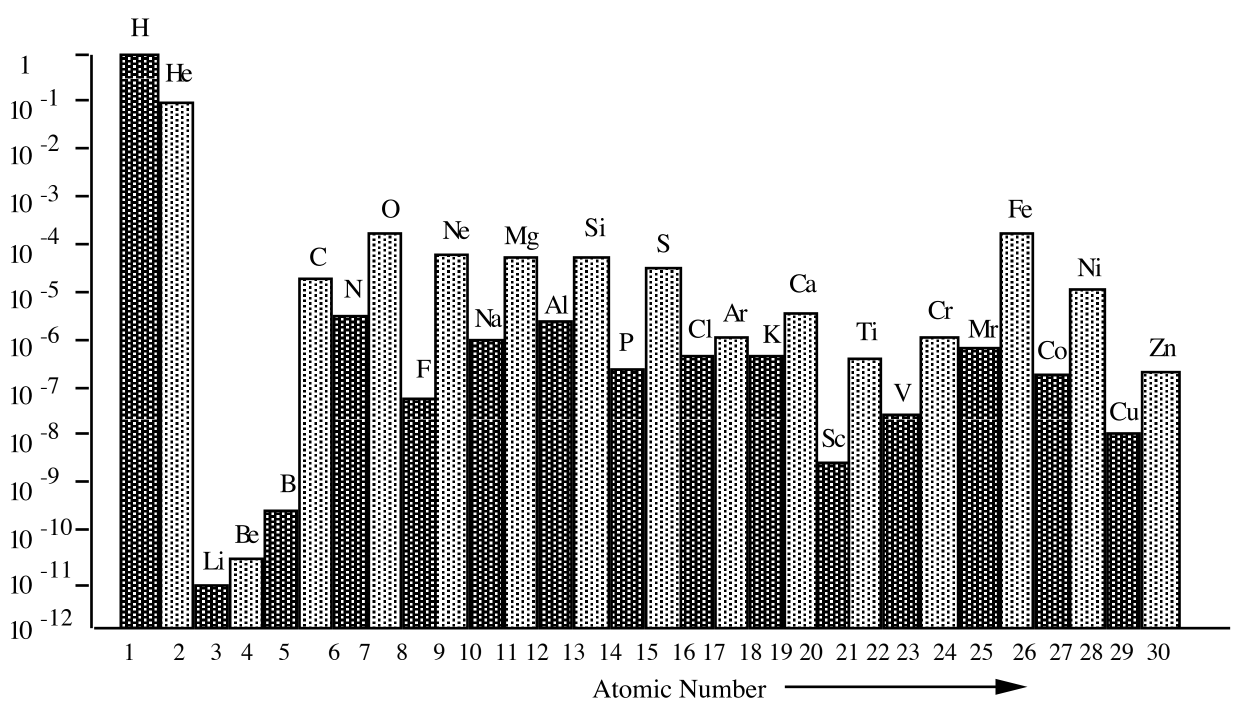
The chemistry is determined by the charge Z and is (almost) independent of A. The Periodic Table classifies the elements into groups. The columns show elements with similar chemical behavior. Thus the alkali metals Li, Na, K, Rb, Cs, Fr and the inert gases He, Ne, A, Kr, Xe, Ra.

Observations of spectra show that stars can be divided into two major classes, called Population I (Pop One) and Population II (Pop Two). Pop I stars have chemical abundances relative to hydrogen similar to the Sun and are young stars, created by ongoing star-formation within the *Galaxy*—(the Galaxy is our galaxy, the Milky Way) and other galaxies. Population II stars are stars with much lower relative abundances of heavier elements—to astronomers "heavy" means beyond helium in the Periodic Table. Astronomers also often call all heavy elements "metals" so the literature must be read with care). The Pop II stars are old stars formed when the heavy element abundances were low. They may be fossils of the initial epoch of star formation after the Big Bang.

The element abundances of Pop I stars can be obtained from observing the nearest example, the Sun, and also, except that H and He have escaped, from measurements of the composition of the Earth itself.

Pop I Abundances

	Atomic Number	Mass (main isotope)	Relative Number	Mass Fraction	
H	1	1	1	0.77	
He	2	4	7×10^{-2}	0.21	
C	6	12	4×10^{-4}	4×10^{-3}	"metals" total 0.02.
N	7	14	9×10^{-5}	1×10^{-3}	
O	8	16	7×10^{-4}	9×10^{-3}	
Ne	10	20	1×10^{-4}	1×10^{-3}	
Mg	12	24	4×10^{-5}	8×10^{-4}	
Si	14	28	4×10^{-5}	8×10^{-4}	
Fe	26	56	3×10^{-5}	1×10^{-3}	



Here is a list of Pop I abundances and a diagram of solar abundances on a logarithmic scale. (Abundances are often presented relative to a hydrogen abundance of 10^{12} on a logarithmic scale—e.g. the abundance of carbon would be given as 8.6). Thus

$$\log \frac{n(\mathrm{C})}{n(\mathrm{H})} = 8.6 - 12 = -3.4$$

$$\frac{n(\mathrm{C})}{n(\mathrm{H})} = 10^{-3.4} = 4 \times 10^{-4} .$$

The helium was produced by nucleosynthesis from the protons and neutrons in the Big Bang (and the amount of helium produced then depended on the number of types of light neutrinos). The abundances of Li, Be and B produced in the Big Bang were very small ($< 10^{-10}$) and they have subsequently been destroyed in stars by nuclear processes. All heavier elements—carbon and beyond—are made only in stars. The combined mass fraction of heavy elements is usually denoted Z (not to be confused with the nuclear charge). Even elements with equal numbers of protons and neutrons tend to be more abundant than odd elements—because a major building block in nucleosynthesis is the ^{4}He nucleus—α-particles—consisting of two protons and two neutrons.

Pop I stars have $Z \sim 0.02$ and are made of material with a heavy element abundance that has been enriched by processing in earlier generations of stars (which have distributed their material back into the interstellar medium through various ejection events, including supernovae).

In contrast Pop II stars which have Z as low or lower than 0.002 may comprise the survivors of the original generation of stars. (There may have been a still earlier generation, referred to as Pop III).

Our galaxy has the shape of a thin disk (thickness ~ 200 pc, radius ~ 8 kpc) with a bulge and a halo of stars.

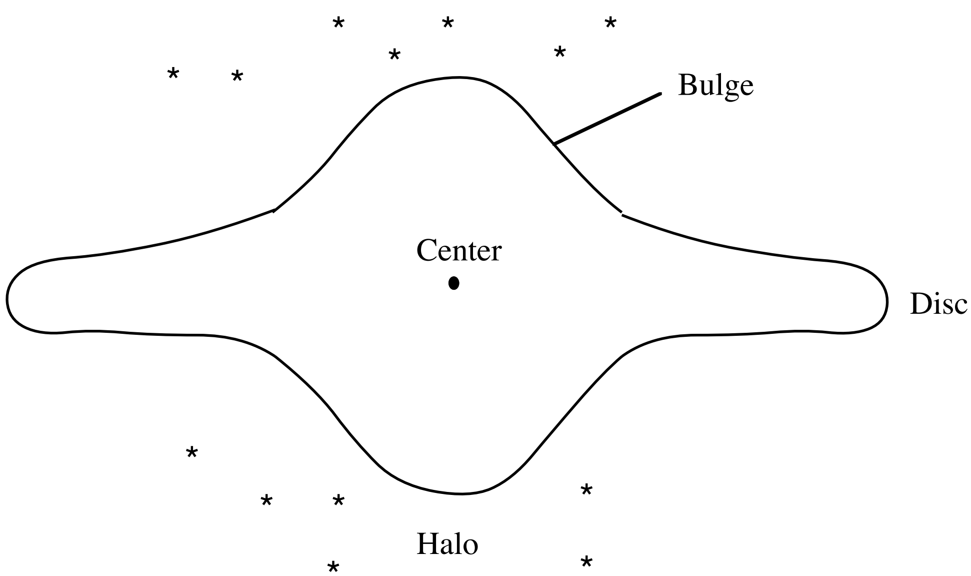


Fig. 5.1

Pop I stars are located mostly in the disk and Pop II stars in the bulge and halo.

5.1.2 Nuclear reactions

Nuclear reactions involve protons, neutrons, positrons, neutrinos and photons (γ-rays). Electrons, muons and neutrinos are leptons (light particles) with lepton number +1. Positrons, anti-muons and antineutrinos have lepton number -1. Electrons and anti-muons are negatively charged, positrons and muons are positively charged. Neutrinos and antineutrinos are neutral. In nuclear reactions, electric charge, atomic mass number and lepton mass number are conserved.

The stars are powered by nuclear reactions that transmute (or burn) lighter elements into heavier elements. The energy we receive as starlight originates in nuclear reactions.

Main sequence stars are powered by the conversion of four hydrogen nuclei-protons—into one helium nucleus-alpha-particle. The process releases energy because the ^{4}He nuclei weighs less than four protons. In atomic mass units (AMU; by definition ^{12}C has a mass of 12 AMU), M_H = 1.0078 and M_{He} = 4.0026. Thus $4M_H - M_{He} = 0.0286$ AMU and 0.71% of the mass of each proton is converted to energy ($E=mc^2$). Unit atomic weight (1 AMU) is 1.66×10^{-27} kg, so energy released per helium nucleus (α-particle) formed is

$$0.0286(1.66 \times 10^{-27})\ (9 \times 10^{16}) \text{ Joules}$$

$$= 4.3 \times 10^{-12} \text{ J} = 4.3 \times 10^{-5} \text{ ergs}$$

$$(1\text{J} = 10^7 \text{ ergs})$$

In the Sun, only in the $0.1M_\odot$ core is the temperature and pressure high enough for the fusion reactions to proceed. So the total thermonuclear energy available is

$$0.0071\ (9 \times 10^{16})\ (0.1M_\odot)$$
$$= 10^{44}\ \text{J} = 10^{51}\ \text{ergs} \quad .$$

Present solar luminosity is 3.90×10^{33} erg s^{-1}, so the Sun will be sustained for 10 billion years, twice its present age.

The actual reaction sequence is the *p-p* (proton-proton) cycle which dominates nucleosynthesis in lower mass stars ($M < 1.5\ M_\odot$) It is the sequence of two-body reactions

$$^1\text{H} + {}^1\text{H} \rightarrow {}^2\text{D} + e^+ + \nu \quad (1.44\ \text{MeV})$$
$$^2\text{D} + {}^1\text{H} \rightarrow {}^3\text{He} + \gamma \quad (5.49\text{MeV})$$
$$^3\text{He} + {}^3\text{He} \rightarrow {}^4\text{He} + {}^1\text{H} + {}^1\text{H} \quad (12.9\ \text{MeV}) \quad .$$

The γ photon is needed in the second reaction to conserve momentum. The neutrino ν is needed in the first reaction to conserve lepton number. You can check that A is conserved by adding the prefixes on each nucleus. The energies in parentheses are the energies released in the reaction. The energy gained by the neutrino escapes. The rest of the energies are converted into thermal energy of the star.

Further reactions occur: ^{7}Be is made from

$$^{3}\mathrm{He} + {}^{4}\mathrm{He} \rightarrow {}^{7}\mathrm{Be} + \gamma \quad .$$

The sequence

$$^{1}\mathrm{H} + {}^{7}\mathrm{Be} \rightarrow {}^{8}\mathrm{B} + \gamma$$

$$^{8}\mathrm{B} \rightarrow {}^{8}\mathrm{Be} + e^{+} + \nu$$

yields a neutrino escaping with energy 7.2 MeV. The measured neutrino flux is ~ half or less than (reliable) solar models predict—this is the solar neutrino problem.

Higher mass stars (> 1.5 $M_{\odot}$) burn via the so-called CNO cycle in which hydrogen is again converted to helium but with carbon acting as a catalyst. The main reactions are

$$^{12}\mathrm{C} + {}^{1}\mathrm{H} \rightarrow {}^{13}\mathrm{N} + \gamma$$

followed by the spontaneous decay of ^{13}N (the stable isotopes of nitrogen are ^{14}N

and ^{15}N)

$$^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^{+} + \nu \quad .$$

Then

$$^{13}\text{C} + {}^{1}\text{H} \rightarrow {}^{14}\text{N} + \gamma$$

$$^{14}\text{N} + {}^{1}\text{H} \rightarrow {}^{15}\text{O} + \gamma$$

(^{16}O, ^{17}O, ^{18}O are the stable isotopes of O)

$$^{15}\text{O} \rightarrow {}^{15}\text{N} + e^{+} + \nu$$

$$^{15}\text{N} + {}^{1}\text{H} \rightarrow {}^{12}\text{C} + {}^{4}\text{He} \; .$$

The ^{12}C is recovered and ^{1}H is converted to ^{4}He. Overall

$$4 \times {}^{1}\text{H} \rightarrow {}^{4}\text{He} + 2\text{e}^{+} + 2\nu$$

plus γ rays.

Thermal fusion reactions make the elements up to Fe. Beyond iron requires an input of energy, say, in supernova explosions.

Because of the Coulomb repulsion, nuclear reaction rates are extremely sensitive to temperature. Above some threshold energy, a small increase in temperature causes a large increase in the reaction rate. The temperature is determined by a balance of the heating and cooling rates. The cooling rate increases exponentially with T. Stars are accordingly thermally stable—the central temperatures where the nuclear reactions occur vary across a wide range of stellar masses in the narrow temperature range 1×10^7K - 2×10^7 K. The temperature comes from a balance between heating and cooling. Cooling reactions increase rapidly in efficiency as T increases between 1 and 2×10^7 K.

A good empirical approximation for the interior or central temperature of main sequence stars is

$$T = 1.5 \times 10^7 \left(\frac{M}{M_\odot}\right)^{1/3} \text{K} \ .$$

5.2 Stellar Structure

5.2.1 Order of magnitude

We can use the virial theorem (p. 4.49) to get an approximate relationship between the mass and radius of a star—the total energy is half the gravitational potential energy.

The potential energy is

$$P.E. = \int \int \frac{Gdm_1 dm_2}{r_{12}} \sim \frac{GM^2}{R}$$

where R is the characteristic size—the radius—and M is the mass.

Now we will show later that

$$K.E. = \frac{3}{2} N_{particles} kT$$

$$\sim \frac{M}{m_p} kT ,$$

m_p is the proton mass.

Equating $P.E.$ and $K.E.$ $GM^2/R = \frac{M}{mp} kT$ and the T-M relationship on the previous page, we relate radius to mass.

$$R \sim \frac{GMm_p}{kT} \sim \frac{GM_\odot m_p}{k(15 \times 10^6 K)} \left(\frac{M}{M_\odot}\right)^{2/3}$$

$$= \frac{(6.67 \times 10^{-8})(1.99 \times 10^{33})(1.67 \times 10^{-24})}{(1.38 \times 10^{-16}) 1.5 \times 10^7} \left(\frac{M}{M_\odot}\right)^{2/3} \text{cm}$$

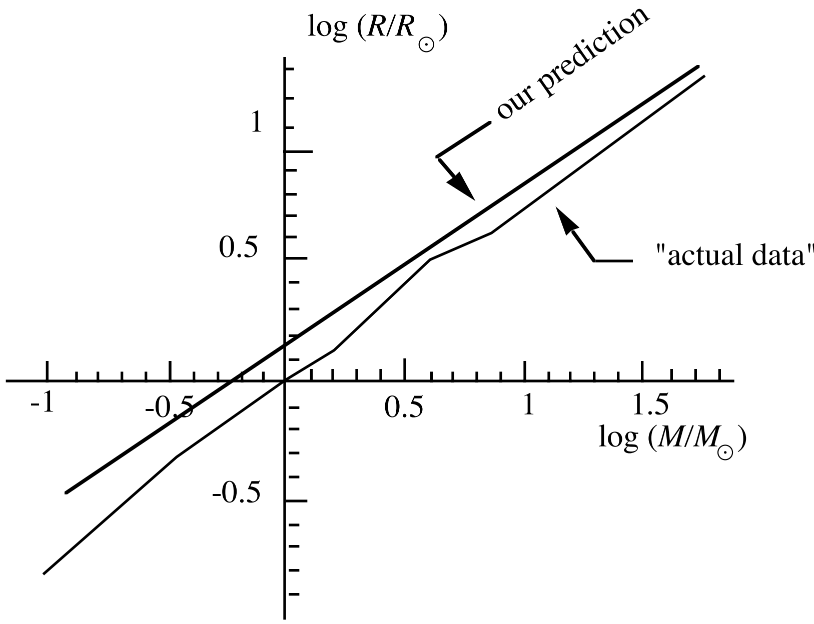
$$= 1.0 \times 10^{11} \left(\frac{M}{M_\odot}\right)^{2/3} \text{cm} .$$

In the case of the Sun, $M = M_\odot$, the actual radius is 7×10^{10} cm.

The Table shows the data for typical main sequence stars

Physical Properties of Main-Sequence Stars

Log ($M/M_\odot$)	Spectral class	Log ($M/M_\odot$)	M_{bol}	M_V	Log ($M/M_\odot$)
-1.0	M6	-2.9	12.1	15.5	-0.9
-0.8	M5	-2.5	10.9	13.9	-0.7
-0.6	M4	-2.0	9.7	12.2	-0.5
-0.4	M2	-1.5	8.4	10.2	-0.3
-0.2	K5	-0.8	6.6	7.5	-0.14
0.0	G2	0.0	4.7	4.8	0.00
0.2	F0	0.8	2.7	2.7	0.10
0.4	A2	1.6	0.7	1.1	0.32
0.6	B8	2.3	-1.1	-2.2	0.49
0.8	B5	3.0	-2.9	-1.1	0.58
1.0	B3	3.7	-4.6	-2.2	0.72
1.2	B0	4.4	-6.3	-3.4	0.86
1.4	08	4.9	-7.6	-4.6	1.00
1.6	05	5.4	-8.9	-5.6	1.15
1.8	04	6.0	-10.2	-6.3	1.3



Mass-Radius Relation for Stars

Fig. 5-2

There is also a mass-luminosity relationship. Very approximately,

$$L \sim M^3$$

for low mass stars but to prove it requires a consideration of the transport of radiation from the interior to the surface.

5.2.2 Stellar interiors

A star is a self-gravitating gaseous system, its interior so hot the material is ionized to nuclei and electrons constituting a fully ionized *plasma*. The quantities needed to specify a stellar interior as a function of radius from the center *r* are

- density $\rho(r)$ - decreasing to zero at the surface. In a gas consisting of n_X particles of mass m_X in unit volume, the total number of particles per unit volume—the particle number density— $n = \sum_X n_X$. The mass density $\rho = \sum n_X m_X$ in units of mass per unit volume—g cm^{-3}.
- mass $M(r)$ - interior to r. At $r = 0$, $M(r) = 0$ *and at the surface* $M(r)$=M, *the mass of the star*
- pressure $P(r)$ - Pressure is force per unit area. Weight is the force created by a gravitational field.The gravitational pressure at r is the total weight per unit area of the overlying mass. It is resisted by the pressure generated by the motions of the particles of the gas.
- temperature T(r) - collisions are frequent and we may assume that at r local thermodynamic equilibrium prevails corresponding to the temperature $T(r)$.
- luminosity L - luminosity is the net outward total energy flow at r. It is zero at the source of its generation ($r \sim 0$) and is constant from there outwards to the surface.
- mean particle mass $\bar{m}(r)$ or mean molecular weight - It is the average mass per unit volume per particle. In terms of it, the gas pressure $P(r) = (\rho/\bar{m}kT = nkT$ where n is the particle density $\left(\bar{m} = \sum_X n_X m_X/n = \rho / n\right)$.
- the opacity $\kappa_\nu(r)$, also called the mass absorption coefficient. It has units

of cm^2 per gram. It is the cross sectional area for absorbing or scattering of photons of frequency ν by a gram of material. Travelling a path length *ds*, the specific intensity I_ν is decreased according to $dI_\nu/ds = -\kappa_\nu \rho I_\nu$.

- nuclear energy generation rate $\varepsilon(r)$ in ergs $cm^{-3}s^{-1}$.

5.2.3 Equations of stellar structure

Assume Hydrostatic Equilibrium.

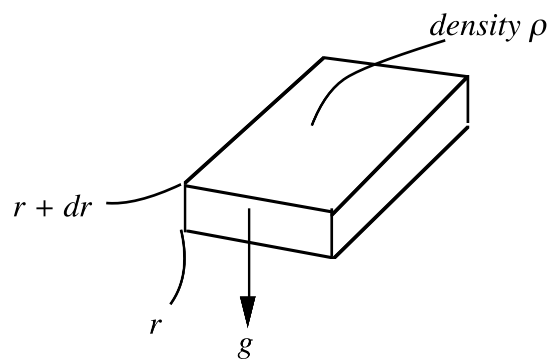


Fig. 5-3

Consider an element of gas between r and $r + dr$ from the center of a star. Pressure $P(r)$ is larger than $P(r + dr)$ by the weight per unit area of the material between r and $r + dr$ in the local gravitational acceleration $g(r)$. If the area of the element is dA, mass of the element is $\rho dA dr$ and weight is $g(r)\rho(r)dAdr = \frac{GM(r)}{r^2}\rho(r)\, dA\ \ dr$ where ρ is the mass density. Thus, the weight of the element per unit area is

$$\frac{G\rho(r)M(r)}{r^2}\ dr.$$

The derivative of the pressure is

$$\frac{dP}{dr} = \frac{P(r+dr) - P(r)}{dr} = -\frac{G\rho(r)M(r)}{r^2}$$

(insert minus sign because pressure increases as r gets smaller). This is the equation of *hydrostatic equilibrium.*

Mass equation

Mass interior to r is

$$M(r) = \int_0^r \rho(r')4\pi r'^2 dr'$$

or equivalently

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho(r)\ .$$

Equation of state

Equation of state is an equation expressing the internal pressure in terms of the density and temperature and it depends on the composition. For a perfect gas,

$P = \frac{\rho}{\bar{m}} kT = nkT$. For a perfect gas in a blackbody radiation field,

$$P = \frac{\rho}{\bar{m}\mu} kT + \frac{1}{3} aT^4 .$$

However in many cases of interest gas pressure substantially exceeds radiation pressure and also μ (i.e. composition) is constant. Then ρ is a function only of T and P can be treated as a function only of ρ.

Polytropes: Lane-Emden equation

We can explore stellar structure by looking at polytropes. A polytropic equation of state is one for which the relationship between P and ρ is a pure power law

$$P = K\rho^{1+1/n}$$

where n, which need not be an integer, is called the polytropic index (the notation arose from a combination of the perfect gas law $P\alpha\rho T$ and an assumed relationship $\rho\alpha T^n$, $T\alpha\ \rho^{1/n}$). Main sequence stars are modeled reasonably by n=3 polytropes (in which case $P = K\rho^{4/3} = \rho \frac{KT}{\bar{m}}$,$T\alpha\ \rho^{1/3}$).

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

or

$$\frac{r^2}{\rho}\ \frac{dP}{dr} = -\ GM$$

mass

$$\frac{dM}{dr} = 4\pi r^2 \rho \quad .$$

Then

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\ \frac{dP}{dr}\right) = -\ \frac{GdM}{dr}$$

$$= -4\pi \mathrm{r}^2 \rho \mathrm{G}$$

and

$$\frac{1}{r^2}\ \frac{d}{dr}\left(\frac{r^2}{\rho}\ \frac{dP}{dr}\right) = -4\pi G\rho \ .$$

Because *P* has been written as a function of ρ, this is an equation for $\rho(r)$ (or $P(r)$.) Remember, we are assuming that

$$P = K\rho^{1+\frac{1}{n}} \quad .$$

which will give us an equation for ρ as a function of r.

Change to dimensionless units

$\xi = \frac{r}{a}$ where a is a scale distance at our disposal

Write $\rho(\mathrm{r}) = \rho_c \phi^n(r)$ where ρ_c is the central density and ϕ is a dimensionless density function. Then

$$\phi(r{=}0) = 1$$

and

$$P = K\rho_c^{\,1+1/\mathrm{n}}\ \phi^{1+n}$$

The equation

$$\frac{1}{r^2}\ \frac{d}{dr}\left(\frac{r^2}{\rho}\ \frac{dP}{dr}\right) = -4\pi G\rho$$

becomes

$$\frac{1}{a^2\xi^2}\ \frac{1}{a}\ \frac{d}{d\xi}\left\{\frac{a^2\xi^2}{\rho_c\phi^n}\ \frac{1}{a}\ \frac{d}{d\xi}\left(K\rho_c{}^{1+1/n}\phi^{1+n}\right)\right\}$$

$$= -4\pi G\rho_c\phi^n$$

which simplifies to

$$\frac{1}{a^2\xi^2}\ \frac{d}{d\xi}\left\{\frac{\xi^2}{\rho_c\phi^n}K\rho_c{}^{1+1/n}(1+n)\,\phi^n\ \frac{d\phi}{d\xi}\right\}$$

$$= -4\pi G\rho_c\phi^n$$

and further to

$$\frac{1}{\xi^2}\ \frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n\left(\frac{4\pi Ga^2}{(1+n)\,\rho_c{}^{-1+\frac{1}{n}}}\ \frac{1}{K}\right).$$

Now choose

$$a^2 = \frac{(1+n)\,\rho_c{}^{\frac{1}{n}-1}K}{4\pi G}\ .$$

We get

$$\frac{1}{\xi^2}\ \frac{d}{d\xi}\left(\xi^2\frac{d\phi}{d\xi}\right) = -\phi^n\ .$$

This is the standard form of the Lane-Emden equation. Its solution yields ϕ as a function of $\xi = r/a$, from which the density and pressure can be obtained as functions of r.

5.2.4 Boundary conditions

It is a second order differential equation and two boundary conditions must be specified, say

$$\phi(\xi = 0) \text{ and } \left.\frac{d\phi}{d\xi}\right|_0 = \phi'(\xi = 0) \ ,$$

using prime to denote differentiation with respect to ξ. Usually the equation must be solved numerically. Write it in the form

$$\phi'' = -\left[\frac{2}{\xi}\phi' + \phi^n\right] .$$

Then, starting at $\xi = 0$ with the specified boundary conditions $\phi(0)$ and $\phi'(0)$, use Taylor series expansions in a stepping sequence

$$\phi(\xi + \Delta\xi) = \phi(\xi) + \phi'(\xi)\Delta\xi$$

and

$$\phi'(\xi + \Delta\xi) = \phi'(\xi) + \phi''(\xi)\Delta\xi$$

$$= \phi'(\xi) - (\tfrac{2}{\xi}\phi' + \phi^n)\Delta\xi \ .$$

Knowing $\phi(\xi + \Delta\xi)$ and $\phi'(\xi + \Delta\xi)$, we can calculate $\phi'(\xi + 2\Delta\xi)$, and so on. The actual boundary conditions are

$$\phi(0) = 1 \quad \text{by definition, } \rho = \rho_c \text{ at } r = 0.$$

To obtain $\phi'(0)$, consider

$$\frac{dP}{dr} = \frac{G\rho M(r)}{r^2} = 0 \text{ at } r = 0. \quad (M(r) \sim r^3)$$

Thus

$$\phi'(0) = 0 \ ,$$

because with $P \sim \rho^{1+1/n} \sim \phi^n$

$$0 = \frac{dP}{dr} \sim \left(1 + \frac{1}{n}\right)\rho^{1/n}\frac{d\rho}{dr} = (n+1)\,\phi^n\frac{d\phi}{dr}$$

$$\sim (n+1)\,\phi^n\phi' \ .$$

ϕ can also be obtained as a power series in ξ , useful for small ξ:

$$\phi(\xi) = \sum_{m=0}^{\infty} c_m \xi^{2m}$$

$$= 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 - \cdots .$$

(The equation is even in ξ. Check $\phi(0) = 1$, $\phi'(0) = 0$.)

The graph shows the results of actual numerical calculations.

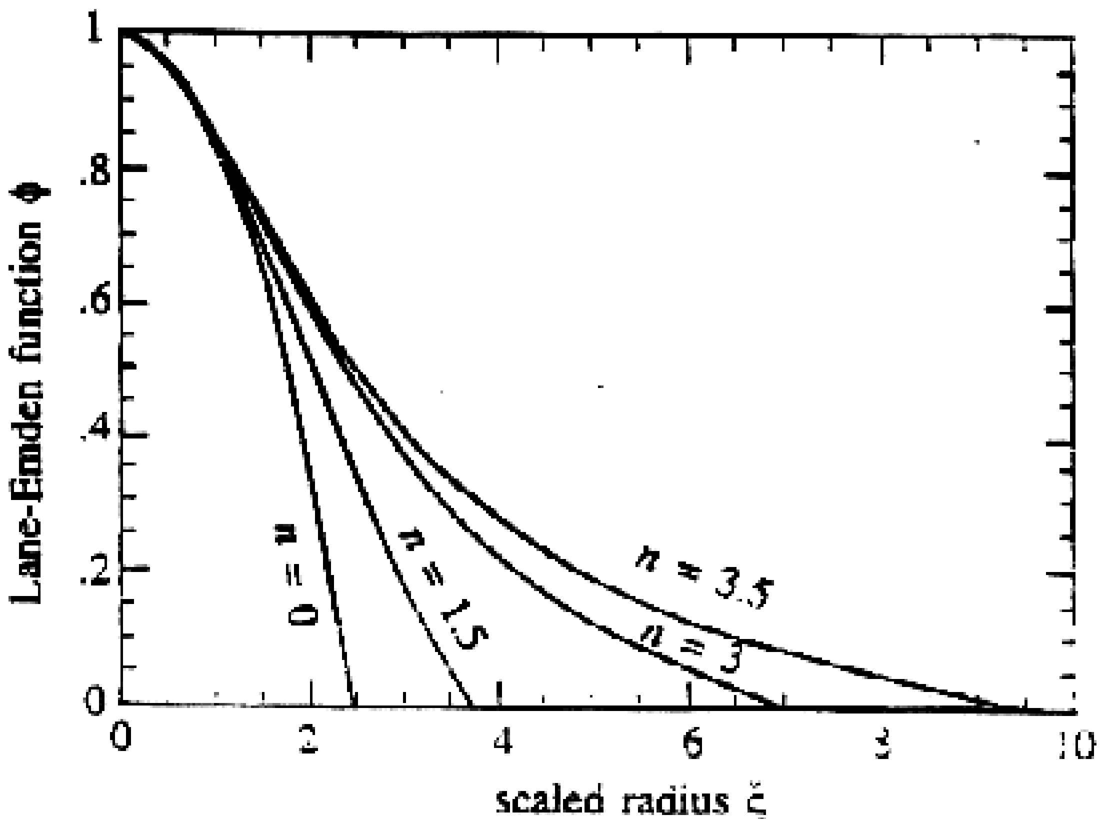


Fig. 5-4

5.2.5 Physical properties of polytropes

For $n < 5$, the Lane-Emden function goes to zero at a finite value of ξ (and hence r) which is denoted ξ_1. It is the surface of the star. The stellar radius is therefore

$$R = a\xi_1 = \sqrt{\frac{(1+n)\,\rho_c^{\frac{1}{n}-1} K}{4\pi G}}\ \xi_1 \ .$$

The mass $M(r)$ is given by

$$M(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'$$

$$= 4\pi a^3 \rho_c \int_0^{\xi = r/a} \xi^2\ \phi^n\ d\xi \ .$$

We can reduce this to a simple form. Multiply the Lane-Emden equation by ξ^2 and integrate from 0 to ξ. We obtain

$$\xi^2\ \frac{d\phi}{d\xi} = -\int_0^{\xi} \xi^2 \phi^n\ d\xi \ .$$

So

$$M(r) = 4\pi a^3 \rho_c \left(-\xi^2 \frac{d\phi}{d\xi} \right)_{\xi = r/a}$$

$$= -\frac{1}{\sqrt{4\pi}} \left[\frac{(n+1)K}{G} \right]^{3/2} \rho_c^{\frac{3-n}{2n}}$$

$$\times \left(\xi^2 \frac{d\phi}{d\xi} \right)_{\xi = r/a} .$$

Total mass is given by putting $\xi = \xi_1$. To get the physical masses and radii of the polytropes we use the table of values obtained by numerical integrations of the Lane-Emden equations.

n	ξ_1	$-\xi_1^2 \left(\frac{d\phi}{d\xi} \right)_{\xi = \xi_1}$	$\rho_c / \bar{\rho}$
0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47

At ξ_1, P=0. Beyond ξ_1, ϕ is negative and non-physical.

In summary the polytropic stellar model gives R and M in terms of ρ_c and K or more generally it gives two algebraic relations between the four quantities K, ρ_c, M and R. Solving the Lane-Emden equation yields pressure and density as a function of radius.

We can derive the mean density $\overline{\rho}$ defined by

$$M = \frac{4}{3}\pi R^3 \; \overline{\rho} \;=\; \frac{4}{3}\pi a^3 \; \xi_1{}^3 \; \overline{\rho}$$

from

$$M(r) = 4\pi \; a^3 \; \rho_c \left(- \; \xi^2 \; \frac{d\phi}{d\xi}\right|_{\xi = r/a}$$

evaluated at ξ_1. Then

$$\frac{\overline{\rho}}{\rho_c} = - \; \frac{3}{\xi_1} \; \frac{d\phi}{d\xi}\bigg|_{\xi = \xi_1} \; .$$

We obtain the central pressure P_c in terms of M and R from

$$P_c = K\rho_c^{\;1 + \frac{1}{n}} \; .$$

Replace ρ_c using

$$R^2 = a^2 \xi_1^{\ 2} = \frac{(1+n)\rho_c^{\frac{1}{n}-1} K}{4\pi G} \ \xi_1^{\ 2}$$

or

$$K = \frac{4\pi G R^2}{(n+1)\,\xi_1^{\ 2}} \ \rho_e^{-\frac{1}{n}+1} \quad .$$

Hence replacing K in $P_c = K\rho_{\mathrm{c}}^{1+1/\mathrm{n}}$

$$P_c = \frac{4\pi G R^2}{(n\ +\ 1)\,\xi_1^{\ 2}} \ \rho_c^{\ 2}$$

Now use $R = a\xi_1$ in expression for M on previous page

$$M = 4\pi a^3 \rho_c\,(-\xi_1^{\ 2}\,\phi_1') \quad (\phi_1' = \phi_1'(\xi_1)\,)$$

$$= 4\pi R^3 \ \rho_c\,(-\frac{1}{\xi_1}\phi_1')\ .$$

So using expression for ρ_c in terms of M and above formula for P_c

$$P_c = \frac{4\pi G R^2}{(n\ +\ 1)} \ \frac{M^2}{16\pi^2 R^6} \ \frac{1}{\phi_1{}'^{\,2}}$$

$$= \frac{1}{4\pi(n\ +\ 1)\left(\phi_1{}'^{\,2}\right)} \ \frac{GM^2}{R^4} \quad .$$

.

Consider an $n = 3$ polytrope and suppose we know the mass $M = 1\ M_\odot$ and the radius R=1 $R_\odot$. With $n = 3$, the Table gives $\xi_1 = 6.90$,

$$-\xi_1^{\ 2}\ \phi_1' = 2.02 \quad , \phi_1' = -0.0424$$

Thus

$$a^3 \rho_c = \frac{-M}{4\pi\xi_1^{\ 2}\phi_1'} = 7.9 \times 10^{31}\text{g} \quad .$$

Now $a = R/\xi_1$ so with $R = 1\ R_\odot$,

$$a = 1.01 \times 10^{10}\ \text{cm.}$$

Then the expression for $a^3\rho_c$ gives us ρ_c.

Therefore if the Sun is an n=3 polytrope,

central density $\rho_c = 70.7$ g cm^{-3}

mean density = 0.0184 ρ_c = 1.41 g cm^{-3}

central pressure $P_c = 1.25 \times 10^{17}$ dyne cm^{-2}

(1 dyne = 1 gm cm s^{-2} = 10^{-5} Newtons)

For the constant K, we obtain $K = 3.85 \times 10^{14}$ cgs

If we assume the perfect gas law, we can determine the temperature from

$$P = nkT = \frac{\rho}{\bar{m}} kT = \frac{\rho}{\mu m_{\mathrm{H}}} kT$$

where n is the particle number density, μ is the mean molecular weight and m_{H} is the mass of a hydrogen atom. The central temperature is determined by M and R given the value of n (here $n = 3$).

$$T_c = \frac{\mu m_{\mathrm{H}} \rho_c}{\rho_c k} = 1.97\mu \times 10^7 \mathrm{K} \ .$$

We show later that μ=0.62.

5.3 The Perfect (Ideal) Gas Law

You may have seen it in the form

$$P\mathsf{V} = NkT$$

or

$$P = nkT$$

where P is the pressure, V the volume of gas, and T the temperature. N is the number of particles and $N/\mathsf{V}=n$ is the number density of particles.

We can, with one assumption, prove it using the same arguments as in §3.1.4 on radiation pressure, except we have particles in place of photons and the energy is $1/2\, m\, \mathsf{v}^2$. Then

$$P = \frac{2}{3} u$$

where u is the energy density. Here the energy density depends upon the distribution of velocities of the particles of the gas $n(\mathrm{v})d\mathrm{v}$:

$$u = \frac{1}{2} \int_0^\infty \mathbf{p} \cdot \mathbf{v} \; n(\mathrm{v})\, d\mathrm{v}$$

and

$$n = \int_0^\infty n(\mathrm{v})\, d\mathrm{v} \; .$$

For non-relativistic particles, $p = m\mathrm{v}$

$$P = \frac{2}{3} u = \frac{1}{3} \int_0^\infty m\,\mathrm{v}^2 \, n(\mathrm{v})\, d\mathrm{v} \; .$$

We now assume we know that $n(\mathrm{v})\, d\mathrm{v}$ is the Maxwell distribution function

$$n(\mathrm{v})\, d\mathrm{v} = n \left(\frac{m}{2\pi kT} \right)^{3/2} e^{-m\mathrm{v}^2/2kT} \; 4\pi\, \mathrm{v}^2 d\mathrm{v} \, .$$

Then $\int n(\mathbf{v})d\mathbf{v} = n$. .

Substituting into the integral for P we obtain

$$P = nkT \quad .$$

The energy density can be given as a mean energy $\overline{E}$ per particle by writing

$$\overline{E} = \frac{1}{n} \int n(\mathrm{v}) E \, d\mathrm{v}$$

or

$$\overline{\mathrm{v}^2} = \frac{1}{n} \int n(\mathrm{v}) \mathrm{v}^2 \, d\mathrm{v}$$

Evaluating the integral, we obtain

$$\overline{\mathrm{v}^2} = \frac{3kT}{m} \quad .$$

Thus, mean kinetic energy per particle is

$$\frac{1}{2} m \overline{\mathrm{v}^2} = \frac{3kT}{2}$$

(each degree of freedom contributes $kT / 2$ to the energy).

If the gas contains particles of different mass, we write $n = \rho / \overline{m}$ where ρ is the mass density and $\overline{m}$ is an average mass often expressed as a *mean molecular* *weight* μ, in units of the mass of the hydrogen atom

$$\mu = \overline{m} / m_{\mathrm{H}} \ ;$$

$m_{\mathrm{H}} = 1.673625 \times 10^{-24}$ g is the mass of a hydrogen atom. Then

$$P = \frac{\rho k T}{\mu m_{\mathrm{H}}} \ .$$

The mean molecular weight depends on composition, and on the state of ionization, because free electrons are included in the definition of $\overline{m}$.

For a neutral gas with N_j particles j and masses $A_j = m_j / m_{\mathrm{H}}$ relative to hydrogen

$$\mu_n m_{\mathrm{H}} = \frac{\sum_j N_j A_j m_{\mathrm{H}}}{\sum_j N_j}$$

$$\text{so} \quad \mu_n = \frac{\sum_j N_j A_j}{\sum_j N_j} .$$

For a completely ionized gas

$$\mu_i = \frac{\sum_j N_j A_j}{\sum_j N_j \left(1 + Z_j\right)} \qquad (\text{electron mass may be ignored})$$

where Z_j is the nuclear charge. (Electron mass is negligible).

Introduce *mass fractions*

$$X = \frac{\rho_{\mathrm{H}}}{\rho} = \frac{\text{total mass of hydrogen}}{\text{total mass}} .$$

Similarly Y for helium and Z for elements heavier than He. Thus $X + Y + Z = 1$.

$$X_j = \frac{\rho_j}{\rho} = \frac{\text{total mass of particle } j}{\text{total mass}} = \frac{N_j m_j}{\sum_j N_j m_j}$$

$$X_{\mathrm{H}} = X , X_{\mathrm{He}} = Y , X_{\mathrm{other}} = Z$$

Then write

$$\frac{1}{\mu_n m_{\mathrm{H}}} = \frac{\sum_j N_j}{\sum_j N_j m_j} = \sum_j \frac{N_j}{N_j m_j} N_j m_j \Big/ \sum_j N_j m_j$$

$$= \sum_j \frac{N_j}{N_j A_j m_{\mathrm{H}}} \left| \frac{N_j m_j}{\sum N_j m_j} \right| = \sum_j \frac{N_j}{N_j A_j m_{\mathrm{H}}} X_j \qquad \left(A_j = A_j m_{\mathrm{H}} \right) .$$

$$= \sum_j \frac{X_j}{A_j} \frac{1}{m_{\mathrm{H}}}$$

So $\dfrac{1}{\mu} = \sum_j X_j / A_j$.

A typical composition for a cosmic gas is

$$X = 0.70 \ , \ Y = 0.28 \ , \ Z = 0.02 \ .$$

The heavy element contribution to μ is small

$$\frac{1}{\mu} = X + \frac{Y}{4} + \left(\frac{\overline{I}}{A}\right) Z \ .$$

$$\left(\frac{\overline{I}}{A}\right) \sim \left(\frac{1}{16}\right)$$

$$\text{so} \quad \frac{1}{\mu_n} = 0.70 + \frac{0.28}{4} + \frac{0.02}{16}$$

$$\mu_n = 1.30 \ .$$

If gas is a fully-ionized plasma,

$$\frac{1}{\mu} = \sum_j \frac{X_j \left(1 + Z_j\right)}{A_j}$$

$$\frac{1}{\mu} = 1.40 + 0.21 + \frac{1}{2}\, 0.02$$

since Z_j = number of electrons = number of protons $\sim \frac{1}{2}$ mass.

Then

$$\mu_i = 0.62 \ .$$

for a fully-ionized cosmic gas.

5.3.1 Adiabatic index

Thermal conductivity is the transfer of kinetic energy (heat) from one particle to another as they collide. If thermal conductivity is negligible and there is no inflow or outflow of heat, pressure of a gas element which is being compressed increases *adiabatically*. In a volume V containing N particles, the total energy E is given by

$$E = \frac{\beta}{2} \ NkT$$

where β is the number of degrees of freedom. (We have just shown that $\beta = 3$ when only translation occurs.) Now use the perfect gas law

$$P = \frac{N}{\mathsf{V}} kT$$

to relate E to P,

$$E = \frac{\beta}{2} \ PV \ .$$

First law of thermodynamics (really conservation of energy)—if you compress volume V to volume V-dV, the internal energy E is increased by the work done in compressing it

$$dE = -P\, d\mathrm{V}$$

(minus sign because internal energy increases with decreasing volume). But

$$dE = \frac{\beta}{2}\left(P d\mathrm{V} + \mathrm{V} dP \right)$$

$$\frac{\beta}{2} \mathrm{V} dP = -\left(1 + \beta/2\right) P d\mathrm{V}$$

$$\frac{dP}{\mathrm{P}} = -\left(\frac{2+\beta}{\beta}\right)\frac{d\mathrm{V}}{\mathrm{V}} \; .$$

Hence

$$P = \text{constant} \times \mathrm{V}^{-(1+2/\beta)}$$

So, for a fixed number of particles in the volume,

$$P \;\alpha\; \rho^{(1+2/\beta)} \;,$$

which is polytropic with index $n = \beta / 2$. For $\beta = 3$

$$n = 1.5 \ , \ P \ \alpha \ \rho^{5/3} \ .$$

As the number of degrees of freedom β increases, the polytropic index increases. The limiting case of an isothermal gas with T constant, $P \alpha \rho T$, means $P \alpha \rho$ which corresponds to $\beta \rightarrow \infty$. The work done is distributed over an infinite number of internal degrees of freedom and the temperature does not change.

As a generalization of the perfect gas law and avoiding the use of temperature, write

$$P = \frac{\gamma' E}{V} \ , E = \frac{PV}{\gamma'}$$

(perfect gas law, $\gamma' = 2/3$).

Then repeating the argument, we have

$$-PdV = dE = \frac{1}{\gamma'}\left(PdV + VdP\right)$$

$$\left(1 + \frac{1}{\gamma'}\right)\frac{dV}{V} = -\frac{1}{\gamma'}\frac{dP}{P}$$

$$P = \text{constant} \times V^{-(\gamma' + 1)}$$

$$\alpha \ \rho^{(\gamma' + 1)} \ .$$

We get a polytropic equation of state. If $\gamma = \gamma' + 1$,

$$P \propto \rho^{\gamma} \equiv \rho^{1+1/n}$$

with

$$\gamma' = \gamma - 1 = \frac{1}{n} \ .$$

So $P \propto \rho^{\gamma}$ implies that a polytropic gas satisfies

$$PV = (\gamma - 1)E \ .$$

γ is the adiabatic index. We will use this later.

5.3.2 Convection

There are three ways of transporting energy or heat: radiation—photons moving in straight lines carry energy—the photons can be absorbed by atoms and then emit isotropically so changing direction or they can be elastically scattered—the path followed by a photon before it finally escapes from the interior through the visible surface can be long. Most of the scattering occurs in the outer layers of the stellar atmosphere—in the interior, the material is fully ionized and scattering and absorption are minimized.

Conduction of heat occurs by collisional transfer of energy with the faster particle giving energy to the slower. In stellar interiors radiation transport by conduction is not very important because the densities are not that high.

The third mechanism is *convection*. It involves actual motion of the gas. Bubbles of gas that are hotter than their surroundings rise into a region of lower pressure where they expand and cool. The bubbles may merge with the surrounding gas or they may fall back down. Mass is conserved so mass going up is replaced by mass going down. "Hot air rises and cool air sinks."

Convection is a complex, not fully understood process. It is usually turbulent and different fluid elements are readily mixed. In gas in turbulent convection, the gas behaves adiabatically (no heat flowing into or out of a volume element) so for β=3

$$P = \text{constant} \times \Box\rho^{5/3}$$

and the constant is the same for all fluid elements. This creates a polytropic model with $n = 3/2$.

The Sun is not fully convective. Most of it is stably stratified with the deeper, denser material having a lower adiabat (smaller constant in the P-ρ relation). For the Sun, $n = 3$ is a good approximation and

$$P \sim \rho^{4/3} \text{ and } T \sim \rho^{1/3} .$$

Sun is convective in the outer one sixth of its atmosphere, whereas low mass stars $M \leq 0.3\ M_{\odot}$ are fully convective throughout and stars of all masses pass through a fully convective phase (called the Hayashi phase) as they evolve on to the main sequence.

5.3.3 Equation of State for Degenerate Matter

Degenerate matter resists compression not by the pressure generated by thermal motion but by the Pauli exclusion principle which asserts that two identical fermions cannot occupy the same quantum state.

The uncertainty principle of quantum mechanics asserts that $\Delta x\ \Delta p \sim h$ or $\Delta \mathbf{x}\ \Delta \mathbf{p}\ \sim h^3$. If electrons are forced into a smaller volume $\Delta \mathbf{x}$, they must be pushed into a larger momentum volume $\Delta \mathbf{p}$. In the lowest energy state of an electron gas, the electrons fill all the momentum states in a sphere out to some radius p_F, called the Fermi momentum.

The volume occupied in momentum space is

$$\frac{4\pi}{3} p_F^{\,3} = \Delta \mathbf{p}\ .$$

If V is the volume in position space,

$$\Delta \boldsymbol{x} = \mathsf{V}$$

$$\Delta \boldsymbol{x} \Delta \boldsymbol{p} = \frac{4\pi}{3} p_F^{\,3}\ \mathsf{V}\ .$$

If N is the total number of electrons, with electron density $n_e = N/\mathsf{V}$, we have for the phase space density $2/h^3$ (in quantum mechanics h^3 is the smallest volume and electrons have two spin states—maximum density = $2/h^3$),

$$\frac{2}{h^3} = \frac{N}{\left(\frac{4\pi}{3} p_F{}^3\right) \mathsf{V}} = \frac{n_e}{\frac{4\pi}{3} p_F{}^3}$$

so

$$p_F{}^3 = \frac{3}{8\pi} h^3 n_e.$$

The energy per unit volume E/V is given by the integral of the electron energy $p^2/2m_e$ over momentum space. The number of electrons with momentum p in phase space is $\frac{2}{h^3}\mathsf{V}\Delta\mathbf{p}$ so number per unit volume of position space is $n_e = \frac{2}{h^3}\ \Delta\mathbf{p}$. Hence

$$\frac{E}{\mathsf{V}} = \int_0^{P_F} \frac{p^2}{2m_e} \frac{2}{h^3}\ 4\pi\ p^2 dp$$

$$= \frac{4\pi}{5} p_F{}^5 / h^3 m_e.$$

The pressure

$$P = \frac{2}{3}\frac{E}{\mathsf{V}}$$

which is proportional to $\rho^{5/3}$ (using $P\mathsf{V} = (\gamma - 1)E$).

Explicitly

$$P = \frac{2}{3}\left(\frac{4\pi p_F^{\,5}}{5}\right)\frac{1}{h^3 m_e}$$

$$= \frac{2}{3}\left(\frac{4\pi}{5}\right)\left(\frac{3}{8\pi}\,h^3 n_e\right)^{5/3}\frac{1}{h^3 m_e}$$

$$= \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e}\,n_e^{\,5/3} \;.$$

This is for non-relativistic electrons. If the gas is so compressed that p_F approaches $m_e c$ (i.e. v approaches c), then as for photons

$$P = \frac{1}{3}\frac{E}{\mathsf{V}} \rightarrow \gamma' = \frac{1}{3}\;,\; \gamma = \frac{4}{3}\;,$$

$$\rightarrow P \alpha\, \rho^{4/3}$$

For relativistic electrons, $E = pc$.

$$\frac{E}{V} = \int_0^{p_F} pc \; \frac{2}{h^3} \; 4\pi p^2 \; dp$$

$$= \frac{2\pi c}{h^3} \; p_F{}^4 \; .$$

Then

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e{}^{4/3} \; .$$

The crossover from non-relativistic to relativistic occurs when the two regimes are about equal

$$\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e{}^{4/3} \sim \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e} \; n_e{}^{5/3}$$

$$\text{i.e.} \quad n_e \sim \frac{4}{3}\left(\frac{2}{5} \; \frac{h}{m_e c}\right)^{-3}$$

($h/m_e c$ is the *Compton* wavelength).

To relate electron density n_e to ρ, in the absence of hydrogen (which occurs for white dwarfs—all the hydrogen and helium have been burnt to carbon and oxygen)

$$\rho = n_e\,(m_e + m_p + m_n) \sim 2\,n_e m_p\ .$$

Each electron is accompanied by one proton and one neutron.

Or write $\rho = \mu_e m_p n_e$ where μ_e is the mean molecular weight per electron. Then, extending to include a possible contribution from hydrogen,

$$\mu_e = \frac{2}{1 + X}$$

where X is mass fraction of mass H.

5.3.4 White Dwarf Stars

White dwarf stars are supported entirely by degeneracy pressure, so we may write

non-relativistic

$$P = \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e m_{\mathrm{H}}^{5/3}\mu_e^{5/3}}\right]\rho^{5/3}$$

relativistic

$$P = \left[\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{m_{\mathrm{H}}^{4/3}\mu_e^{4/3}}\right]\rho^{4/3} \quad .$$

These expressions determine the constants K in the formula

$$P = K\rho^{1+1/n}$$

with $n = \dfrac{3}{2}$ for the non-relativistic and $n = 3$ for the relativistic case. Then, using 5.26, white dwarf radius R and mass M are given by

$$R = \xi_1 \left[\frac{(n+1)K_e}{4\pi G}\right]^{1/2} \rho_c^{(1-n)/2n}$$

$$M = -\xi_1^2 \phi_1{}' \, 4\pi \left[\frac{(n+1)K_e}{4\pi G}\right]^{3/2} \rho_c^{(3-n)/2n} \quad .$$

Non-relativistic White Dwarf

$n = \dfrac{3}{2}$, $\xi_1 = 3.654$, $-\xi_1^2\phi_1{}' = 2.714$

$$R = (1.22 \times 10^4 \text{ km})\,(\rho_c/10^6 \text{g cm}^{-3})^{-1/6}\left(\frac{\mu_e}{2}\right)^{-5/6}$$

$$M = (10.4964\, M_\odot)\,(\rho_c/10^6 \text{g cm}^{-3})^{1/2}\left(\frac{\mu_e}{2}\right)^{-5/2}$$

($\mu_e \sim 2$ for a white dwarf which consists mostly of C and O).

Eliminating unknown central density,

$$M = (0.7011\ M_{\odot})(R/10^4\ \text{km})^{-3}\left(\frac{\mu_e}{2}\right)^{-5} .$$

So $M \sim R^{-3}$. The more massive a white dwarf, the smaller it is.

Relativistic White Dwarf

n=3 , ξ_1 = 6.897 , $-\xi_1\phi_1'$ = 2.018 ,

$$R = (3.347\times 10^4\ \text{km})\ (\rho_c/10^6\ \text{g cm}^{-3})^{-1/3}\left(\frac{\mu_e}{2}\right)^{-2/3}$$

$$M = (1.4567\ M_{\odot})\left(\frac{2}{\mu_e}\right)^2 .$$

M is independent of ρ_c and has a fixed value 1.457 $M_{\odot}$ (called the Chandrasekhar mass). 1.457 $M_{\odot}$ is the relativistic limit.

Fig. 5-5 reproduces observational data showing mass M as a function of radius R (The theory works well!)

(In terms of fundamental constants, the Chandrasekhar mass is

$$M_{ch} = 3.10\left(\frac{\hbar c}{G}\right)^{3/2}\frac{1}{m_H^{\,2}\ \mu_e^{\,2}} .$$

$\left(\frac{\hbar c}{G}\right)^{1/2}$ is the *Planck mass* = 0.22 μg .)

5.3.5 Neutron stars

Beyond the Chandrasekhar mass, the electrons are forced to combine with protons to form neutrons and the star becomes a degenerate neutron star which can be analyzed in the same way as white dwarfs with the neutron molecular weight replacing the μ_e. However general relativity plays a role and the physics is still uncertain.

Pulsars are spinning neutron stars, detected by the regular arrival of radio pulses thought to be due to electrons trapped in the magnetic field of the star. The magnetic fields are of the order of 10^{12} Gauss. Limiting mass is about 6 $M_\odot$. Masses beyond 6 $M_\odot$ collapse to form black holes.

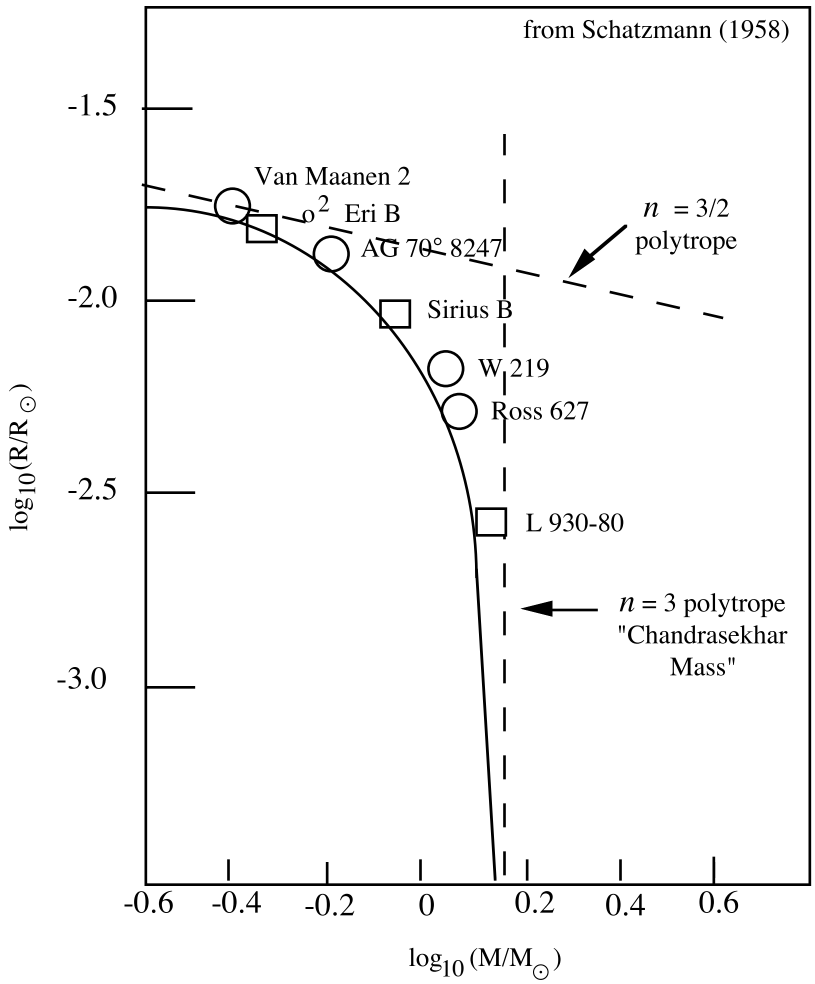

from Schatzmann (1958)
-1.5
-2.0
-2.5
-3.0
Van Maanen 2
o2 Eri B
AG 70° 8247
Sirius B
W 219
Ross 627
L 930-80
n = 3/2
polytrope
n = 3 polytrope
"Chandrasekhar
Mass"
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
log10(R/R☉)
log10(M/M☉)

Fig. 5-5

5.3.6 Black holes

If $M > \sim 6M_\odot$, matter collapses to a black hole when the escape veolocity equals the velocity of light. The star vanishes inside the Schwarzchild radius given by

$$c^2 = 2GM/r_s \,.$$

Numerically

$$r_s \sim 3(M/M_\odot)\ \text{km}\,.$$

5.3.7 Stellar structure virial theorem

We can use the virial theorem to show that stars are unstable if the adiabatic index is less than 4/3.

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\,\rho(r)\,.$$

Let $\mathsf{V}(r)$ be the volume inside a radius r

$$\mathsf{V}(r) = \frac{4}{3}\pi r^3$$

$d\mathsf{V}(r)$ is the volume of the shell between r and $r + dr$ containing a mass $dM(r)$

$$dM(r) = \rho(r) 4\pi r^2 dr$$

Multiply pressure equation by $V(r)\, dr$:

$$V(r) \frac{dP}{dr} dr = - \frac{GM(r)\rho(r)}{r^2} V(r) dr \ .$$

which we can write

$$V(r) dP = - GM(r)\rho(r) \frac{4}{3}\pi r^3 dr \frac{1}{r^2}$$

$$= \frac{1}{3} GM(r)\, dM(r) \frac{1}{r} \ .$$

Integrate over the star of radius R

$$\int_0^R V(r) dP = - \frac{1}{3} \int_0^R \frac{GM(r)\, dM(r)}{r}$$

$$= \frac{1}{3} U$$

where U is the total gravitational potential energy. Integrate by parts to get

$$\frac{1}{3} U = [PV]_0^R - \int_{r=0}^{R} P(r) dV \ .$$

At $r = 0$, $\mathsf{V} = 0$. At $r=R$, $P = 0$.

Hence

$$U + 3 \int_0^R P d\mathsf{V} = 0 \quad .$$

This is the stellar structure virial theorem. For a polytropic gas, energy per unit volume is

$$u = \frac{P}{\gamma - 1} \quad .$$

Thus

$$\int P d\mathsf{V} = (\gamma - 1) \int u d\mathsf{V}$$

$$= (\gamma - 1) E$$

where E is the total *internal* energy of the star.

So

$$U + 3(\gamma - 1) E = 0 \quad .$$

(For a perfect monatomic gas, only translation is possible, $\gamma = 5/3$ and $E = T$.

Then $U + 3(\gamma - 1)E = 0$ asserts that $U + 2T = 0$ which we proved earlier for a system of particles moving under their gravitational attraction.)

The total energy of a star is

$$E_{tot} = E + U$$

$$= E - 3(\gamma - 1)E = -(3\gamma - 4)E$$

$$= \frac{3\gamma - 4}{3(\gamma - 1)} U$$

If $\gamma > \frac{4}{3}$, $E_{tot} < 0$ — star is bound

$\gamma < \frac{4}{3}$, $E_{tot} > 0$ — star is unbound

— it is unstable to the conversion of the internal energy into expansion velocity and it blows itself apart.