$$\frac{1}{\mu_m m_h} = \frac{\sum_j N_j}{\sum_j N_j m_j} = \sum_j \frac{N_j}{N_j m_j} N_j m_j \Big/ \sum_j N_j m_j$$

$$= \sum_j \frac{N_j}{N_j A_j m_H} \left| \frac{N_j m_j}{\sum_j N_j m_j} \right|$$

$$= \sum_j \frac{N_j}{N_j A_j m_H} X_j \quad \left(m_j = A_j m_H\right) \tag{5-53}$$

$$= \sum_j \frac{X_j}{A_j} \frac{1}{m_H} .$$

so

$$\frac{1}{\mu} = \sum_j X_j A_j \tag{5-54}$$

A typical composition for a cosmic gas is

$$X = 0.70 \quad , \quad Y = 0.28 \quad , \quad Z = 0.02 .$$

The heavy element contribution to μ is small

$$\frac{1}{\mu} = X + \frac{Y}{4} + \left(\overline{\frac{1}{A}}\right) Z .$$

$$\left(\overline{\frac{1}{\mathrm{A}}}\right) \sim \left(\frac{1}{16}\right) \tag{5-55}$$

so

$$\frac{1}{\mu_n} = 0.70 + \frac{0.28}{4} + \frac{0.02}{16}$$

$$\mu_n = 1.30 \qquad (5\text{-}56)$$

If gas is a fully-ionized plasma (from 5-50),

$$\frac{1}{\mu} = \sum_j \frac{X_j\left(1 + Z_j\right)}{A_j}$$

$$\frac{1}{\mu} = 1.40 + 0.21 + \frac{1}{2}\ 0.02 \qquad (5\text{-}57)$$

since Z_j = number of electrons = number of protons ~$\frac{1}{2}$ mass.

Then

$$\mu_i = 0.62\ . \qquad (5\text{-}58)$$

for a fully-ionized cosmic gas.

5.4.1 Adiabatic index

Consider now the internal energy of a gas. Thermal conductivity is the transfer of kinetic energy (heat) from one particle to another as they collide. If thermal conductivity is negligible and there is no inflow or outflow of heat, the pressure of a gas element which is being compressed is said to increase *adiabatically*. For a perfect gas the internal energy is

$$E = \frac{3}{2}\ NkT = \frac{3}{2}\ P\,V \qquad (5\text{-}59)$$

Generalize the relationship to

$$E=P\mathsf{V}/\gamma' \tag{5-60}$$

The first law of thermodynamics (essentially the conservation of energy) asserts that if you compress a volume from V to $\mathsf{V}-d\mathsf{V}$ the internal energy is increased by the work done.

$$dE=-Pd\mathsf{V} \tag{5-61}$$

From $E = P\mathsf{V}/\gamma'$, we have

$$dE=\frac{1}{\gamma'},\ (Pd\mathsf{V} + \mathsf{V}dP) \tag{5-62}$$

$$\therefore \left(1 + \frac{1}{\gamma'}\right)\frac{d\mathsf{V}}{\mathsf{V}} = \frac{1}{\gamma'}\ \frac{dP}{P}\ . \tag{5-63}$$

So

$$P = \text{constant} \times \mathsf{V}^{-(\gamma' + 1)} \tag{5-64}$$

Therefore for a fixed number of particles in the volume of gas

$$P=K\,\rho^{(\gamma' + 1)} \tag{5-65}$$

where K is a constant. This is the *polytropic* equation of state (5-33). Write $\gamma = \gamma' + 1 \gamma$ is called the adiabatic index. Then

$$P = K\,\rho^{\gamma} = K\,\rho^{1+1/n} \tag{5.66}$$

So

$$\frac{1}{n} = \gamma - 1 \tag{5-67}$$

We can use the argument in reverse. If $P \alpha \rho^{\gamma}$, then

$$PV = (\gamma\text{-}1)\, E. \tag{5-68}$$

The adiabatic index can be related to the number of degrees of freedom and to the specific heats at constant temperature and at constant pressure (see Ostlie and Carroll pp.353-354).

For a perfect gas there are three degrees of freedom,

$$E = \frac{3}{2} NkT$$

$$= \frac{3}{2} \mathrm{P\ V} \tag{5-69}$$

So $\gamma = \frac{5}{3}$, $\mathrm{n} = \frac{3}{2}$.

$$P \alpha \rho^{5/3} . \tag{5-70}$$

In the case of a thermal gas with T independent of ρ, $P \alpha \rho$ and the polytropic index is infinite. It corresponds to an infinite number of degrees of freedom over which the work done is distributed.

5.4.2 Convection

There are three ways of transporting energy or heat: radiation—photons moving in straight lines carry energy—the photons can be absorbed by atoms and then emit isotropically so changing direction or they can be elastically scattered—the path followed by a photon before it finally escapes from the interior through the visible surface can be long. Most of the scattering occurs in the outer layers of the stellar atmosphere—in the interior, the material is fully ionized and scattering and absorption are minimized.

Conduction of heat occurs by collisional transfer of energy with the faster particle giving energy to the slower. In stellar interiors radiation transport by conduction is not very important because the densities are not that high.

The third mechanism is *convection*. It involves actual motion of the gas. Bubbles of gas that are hotter than their surroundings rise into a region of lower pressure where they expand and cool. The bubbles may merge with the surrounding gas or they may fall back down. Mass is conserved so mass going up is replaced by mass going down. “Hot air rises and cool air sinks.”

Convection is a complex, not fully understood process. It is usually turbulent and different fluid elements are readily mixed. In gas in turbulent convection, the gas behaves adiabatically (no heat flowing into or out of a volume element) so for a perfect gas

$$P = \text{constant} \times \rho^{5/3} \qquad (5\text{-}71)$$

and the constant is the same for all fluid elements. This creates a polytropic model with $n = 3/2$.

The Sun is not fully convective. Most of it is stably stratified with the deeper, denser material having a lower adiabat (smaller constant in the P-ρ relation). For the Sun, $n = 3$ is a good approximation and

$$P \sim \rho^{4/3} \text{ and } T \sim \rho^{1/3} . \tag{5-72}$$

Sun is convective in the outer one sixth of its atmosphere, whereas low mass stars $M \leq 0.3\ M_\odot$ are fully convective throughout and stars of all masses pass through a fully convective phase (called the Hayashi phase) as they evolve on to the main sequence.

5.4.3 Equation of State for Degenerate Matter

Degenerate matter resists compression not by the pressure generated by thermal motion but by the Pauli exclusion principle which asserts that two identical fermions cannot occupy the same quantum state.

The uncertainty principle of quantum mechanics asserts that $\Delta x\, \Delta p \sim h$ or $\Delta \mathbf{x}\, \Delta \mathbf{p} \sim h^3$. If electrons are forced into a smaller volume $\Delta \mathbf{x}$, they must be pushed into a larger momentum volume $\Delta \mathbf{p}$. In the lowest energy state of an electron gas, the electrons fill all the momentum states in a sphere out to some radius p_F, called the Fermi momentum.

The volume occupied in momentum space is

$$\frac{4\pi}{3} p_F^{\,3} = \Delta \mathbf{p} . \tag{5-73}$$

If V is the volume in position space,

$$\Delta\mathbf{x} = \mathsf{V}$$

$$\Delta\mathbf{x}\Delta\mathbf{p} = \frac{4\pi}{3}\, p_F{}^3\, \mathsf{V}\,. \qquad (5\text{-}74)$$

The product of momentum space and position space is called phase space. In quantum mechanics, the maximum phase space density is $2/h^3$ (h^3 is the smallest allowed volume and electrons have two spin states.

If N is the total number of electrons,

$$\frac{2}{h^3} = \frac{N}{\Delta\mathbf{x}\Delta\mathbf{p}} = \frac{N}{\left(\frac{4\pi}{3}p_F{}^3\right)\mathsf{V}}\,. \qquad (5\text{-}75)$$

But $\frac{N}{\mathsf{V}}$ = electron density n_e so

$$p_F{}^3 = \frac{3}{8\pi}h^3 n_e. \qquad (5\text{-}76)$$

The energy per unit volume E/V is given by the integral of the electron energy $p^2/2m_e$ over momentum space. The number of electrons with momentum of magnitude p in phase space is $\frac{2}{h^3}\mathsf{V}\Delta p$ so number per unit volume of position space is $n_e = \frac{2}{h^3}\,\Delta p$. Hence

$$\frac{E}{V} = \int_0^{p_F} \frac{p^2}{2m_e} \frac{2}{h^3} \Delta\mathbf{p} = \int_0^{p_F} \frac{p^2}{2m_e} \frac{2}{h^3} 4\pi p^2 \, dp \tag{5-77}$$

$$= \frac{4\pi}{5} p_F{}^5 / h^3 m_e .$$

The pressure

$$P = \frac{2}{3} \frac{E}{V} \tag{5-78}$$

which is proportional to $\rho^{5/3}$ (using $PV = (\gamma - 1)E$) (eqn 5-68).

Explicitly

$$P = \frac{2}{3} \left(\frac{4\pi p_F{}^5}{5} \right) \frac{1}{h^3 m_e}$$

$$= \frac{2}{3} \left(\frac{4\pi}{5} \right) \left(\frac{3}{8\pi} h^3 n_e \right)^{5/3} \frac{1}{h^3 m_e} \tag{5-79}$$

$$= \frac{1}{20} \left(\frac{3}{\pi} \right)^{2/3} \frac{h^2}{m_e} n_e{}^{5/3} .$$

This is for non-relativistic electrons. If the gas is so compressed that p_F approaches $m_e c$ (i.e. v approaches c), then as for photons

$$P = \frac{1}{3}\frac{E}{V} \rightarrow \gamma' = \frac{1}{3}\ ,\ \gamma = \frac{4}{3}\ , \qquad (5\text{-}80)$$

$$\rightarrow P \alpha \rho^{4/3}$$

For relativistic electrons, energy of an individual electron is *pc,* so

$$\frac{E}{V} = \int_0^{p_F} pc\, \frac{2}{h^3}\, 4\pi p^2\, dp \qquad (5\text{-}81)$$

$$= \frac{2\pi c}{h^3}\, p_F^{\ 4}\ .$$

Then

$$P = \frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{\ 4/3}\ . \qquad (5\text{-}82)$$

The crossover from non-relativistic to relativistic occurs when the two regimes are about equal

$$\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3} hcn_e^{\ 4/3} \sim \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e}\, n_e^{\ 5/3} \qquad (5\text{-}83)$$

$$\text{i.e.} \quad n_e \sim \frac{4}{3}\left(\frac{2}{5}\ \frac{h}{m_e c}\right)^{-3}$$

(h/m_ec is the *Compton* wavelength).

To relate electron density n_e to ρ, in the absence of hydrogen (which occurs for white dwarfs—all the hydrogen and helium have been burnt to carbon and oxygen)

$$\rho = n_e\,(m_e + m_p + m_n) \sim 2\,n_e m_p \sim 2 n_e m_{\mathrm{H}}\ . \qquad (5\text{-}84)$$

Each electron is accompanied by one proton and one neutron.

Or write $\rho = \mu_e m_{\mathrm{H}} n_e$ where μ_e is the mean molecular weight per electron. Then, extending to include a possible contribution from hydrogen,

$$\mu_e = \frac{2}{1 + X} \qquad (5\text{-}85)$$

where X is mass fraction of mass H(cf. 5-54).

5.4.4 White Dwarf Stars

White dwarf stars are supported entirely by degeneracy pressure, so we may write:

non-relativistic

$$P = \left[\frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{m_e m_{\mathrm{H}}{}^{5/3}\mu_e{}^{5/3}}\right]\rho^{5/3} \qquad (5\text{-}86)$$

relativistic

$$P = \left[\frac{1}{8}\left(\frac{3}{\pi}\right)^{1/3}\frac{hc}{m_{\mathrm{H}}{}^{4/3}\mu_e{}^{4/3}}\right]\rho^{4/3}\ . \qquad (5\text{-}87)$$

Recall the approximate expression (5-26)

$$P_c = \frac{2}{3}\pi G\rho^2 R^2 \qquad (5\text{-}88)$$

and put it equal to the degeneracy pressure. For a non-relativistic white dwarf

$$\frac{2}{3}\pi G\rho^2 R^2 = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3}\frac{\hbar^2}{m_e m_h{}^{5/3}\mu_e{}^{5/3}}\rho^{5/3}\ . \qquad (5\text{-}89)$$

So $R^2\ \alpha\ \rho^{-1/3}$, $R^6\ \alpha\ \rho^{-1}$. Now $\rho = M/\frac{4}{3}\pi R^3$ .and hence $MR^3\ \alpha\ R^6\rho$ = constant. So the more massive the white dwarf, the smaller it is.

To support a more massive star through degeneracy pressure, the more densely packed must be the electrons.

For a $1M_\odot$ white dwarf star of carbon and oxygen, R ~ 6×10^8 cm. (For the Sun, $R_\odot = 7 \times 10^5$ km = 7×10^{10} cm).

There is a maximum mass for white dwarfs because the electron velocities must be less than c and we must use the relativistic pressure (5-87). Then P is proportional to $\rho^{4/3}$. Thus $R^2\ \alpha\ \rho^{-2/3}$, $R\ \alpha\ \rho^{-1/3}$ and $M = \frac{4}{3}\pi R^3\rho$. It equals

1.457 $M_\odot$—this is the Chandrasekhar limit. No white dwarf has been discovered with a mass exceeding the Chandrasekhar limit.

Fig. 5-5 reproduces observational data showing mass M as a function of radius R (The theory works well!)

(In terms of fundamental constants, the Chandrasekhar mass is

$$M_{ch} = 3.10 \left(\frac{\hbar c}{G} \right)^{3/2} \frac{1}{m_H^{\,2}\, \mu_e^{\,2}} \ . \qquad \text{(5-90)}$$

$\left(\frac{\hbar c}{G} \right)^{1/2}$ is the *Planck mass* = 0.22 μg .)

5.4.5 Neutron stars

Beyond the Chandrasekhar mass, the electrons are forced to combine with protons to form neutrons and the star becomes a degenerate neutron star which can be analyzed in the same way as white dwarfs with the neutron molecular weight replacing the μ_e. However general relativity plays a role and the physics is still uncertain.

Pulsars are spinning neutron stars, detected by the regular arrival of radio pulses thought to be due to electrons trapped in the magnetic field of the star. The magnetic fields are of the order of 10^{12} Gauss. Limiting mass is about 6 $M_\odot$. Masses beyond 6 $M_\odot$ collapse to form black holes.

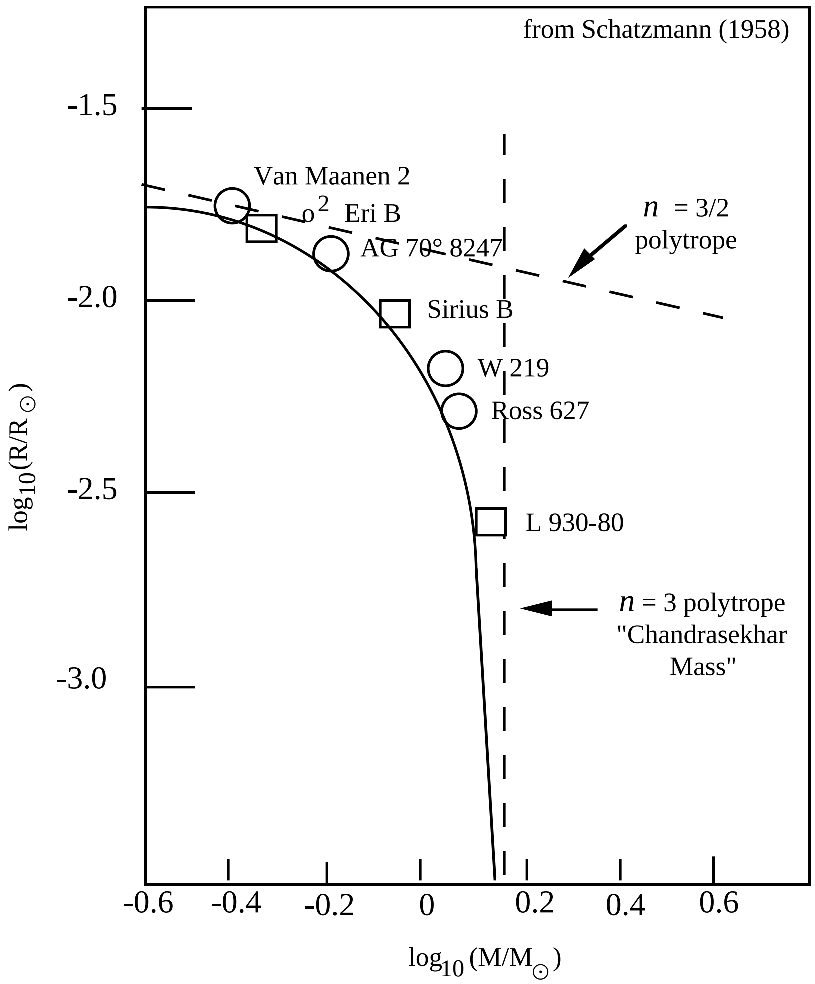
from Schatzmann (1958)
-1.5
-2.0
-2.5
-3.0
Van Maanen 2
o2 Eri B
AG 70° 8247
Sirius B
W 219
Ross 627
L 930-80
n = 3/2
polytrope
n = 3 polytrope
"Chandrasekhar
Mass"
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
log10(R/R☉)
log10 (M/M☉)

Fig. 5-7

5.4.6 Black holes

If $M > {\sim}6M_\odot$, matter collapses to a black hole when the escape veolocity equals the velocity of light. The star vanishes inside the Schwarzchild radius given by

$$c^2 = 2GM/r_s\ . \tag{5-91}$$

Numerically

$$r_s \sim 3(M/M_\odot)\text{ km}\ . \tag{5-92}$$

5.4.7 Stellar structure virial theorem

We can use the virial theorem to show that stars are unstable if the adiabatic index is less than 4/3.

Hydrostatic equilibrium

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2}\rho(r)\ . \tag{5-93}$$

Let $\mathsf{V}(r)$ be the volume inside a radius r

$$\mathsf{V}(r) = \frac{4}{3}\pi r^3 \tag{5-94}$$

$d\mathsf{V}(r)$ is the volume of the shell between r and $r + dr$ containing a mass $dM(r)$

$$dM(r) = \rho(r)4\pi r^2 dr \qquad (5\text{-}95)$$

Multiply pressure equation by V(r) dr:

$$\mathrm{V}(r)\frac{dP}{dr}dr = -\frac{GM(r)\rho(r)}{r^2}\mathrm{V}(r)dr \ . \qquad (5\text{-}96)$$

which we can write

$$\begin{aligned} \mathrm{V}(r)dP &= -GM(r)\rho(r)\frac{4}{3}\pi r^3 dr\frac{1}{r^2} \\ &= \frac{1}{3}GM(r)dM(r)\frac{1}{r} \ . \end{aligned} \qquad (5\text{-}97)$$

Integrate over the star of radius R

$$\begin{aligned} \int_0^R \mathrm{V}(r)dP &= -\frac{1}{3}\int_0^R \frac{GM(r)dM(r)}{r} \\ &= \frac{1}{3}U \end{aligned} \qquad (5\text{-}98)$$

where U is the total gravitational potential energy. Integrate by parts to get

$$\frac{1}{3}U = [P\mathrm{V}]_0^R - \int_{r=0}^{R} P(r)d\mathrm{V} \quad . \tag{5-99}$$

At $r = 0$, $\mathrm{V} = 0$. At r=R, $P = 0$.

Hence

$$U + 3\int_0^R Pd\mathrm{V} = 0 \quad . \tag{5-100}$$

This is the stellar structure virial theorem. For a polytropic gas, energy per unit volume is

$$u = \frac{P}{\gamma - 1} \quad . \tag{5-101}$$

Thus

$$\int Pd\mathrm{V} = (\gamma - 1)\int ud\mathrm{V}$$

$$= (\gamma - 1)E \tag{5-102}$$

where E is the total *internal* energy of the star.

So

$$U + 3\,(\gamma - 1)E = 0 \quad . \tag{5-103}$$

(For a perfect monatomic gas, only translation is possible, $\gamma = 5/3$ and $E = T$. Then $U + 3(\gamma - 1)E = 0$ asserts that $U + 2T = 0$ which we proved earlier for a system of particles moving under their gravitational attraction.)

The total energy of a star is

$$
\begin{aligned}
E_{tot} &= E + U \\
&= E - 3(\gamma - 1)E = -(3\gamma - 4)E \\
&= \frac{3\gamma - 4}{3(\gamma - 1)} U
\end{aligned}
\qquad (5\text{-}104)
$$

If $\gamma > \frac{4}{3}$, $E_{tot} < 0$ — star is bound

$\gamma < \frac{4}{3}$, $E_{tot} > 0$ — star is unbound

— it is unstable to the conversion of the internal energy into expansion velocity and it blows itself apart.