6. Cosmology

6.1 Cosmological Principle

Assume Universe is isotropic (same in all directions) and homogeneous (same at all points)—probably true on a sufficiently large scale. The present Universe has a scale size

$$R_H \sim \frac{c}{H_0} \sim h^{-1}\ 3000\ \text{Mpc} \qquad (6.1)$$

where the Hubble constant is written

$$H_0 = 100h\ \text{km s}^{-1}\ \text{Mpc}^{-1}$$
$$(h \sim 0.7) \qquad (6.2)$$

H is obtained from

$$\text{v} = HD \qquad (6.3)$$

where D is distance and v is velocity. H_0 is the value of H today. We show later that the present density $\rho_H \sim 3\, H_0^{\ 2} / 8\pi G = 1.9\ \times 10^{-29}\ h^2\ g\ \text{cm}^{-3}$.

Universe is inhomogeneous on smaller scales

Scale L	Object	Mass	L/R_H	ρ/ρ_H
30 pc	star cluster	$10^6 M_\odot$	10^{-8}	10^9
30 kpc	galaxy	$3\times10^{11} M_\odot$	10^{-5}	$10^{5.5}$
3 Mpc	galaxy cluster	$10^{15} M_\odot$	10^{-3}	10^3
30 Mpc	galaxy supercluster	$10^{16} M_\odot$	10^{-2}	10

As L increases, density contrast decreases so (maybe) Universe is homogeneous for $L \geq 0.01\ R_H$.

6.2 Cosmic Microwave Background Radiation

Blackbody T = 2.73K (see Fig. 3.7).

Energy density $u = aT^4 = 4.23 \times 10^{-13}$ ergs cm^{-3}

Photon density n = 413 cm^{-3}

Measured to be smooth at a level of 10^{-5} so (almost) homogeneous when the radiation was emitted (at $z \sim 1500$, z is red shift, $z = 1 - \frac{\nu}{\nu_0} \sim \frac{\mathrm{v}}{c}$).

There is a dipole anistropy in the measured spectrum caused by the motion of the solar system through the radiation field.

Doppler effect shifts the frequency by $\nu \rightarrow \nu\ (1+\mathrm{v}/c)$ where v=369km s^{-1}. So T is higher in the direction of motion and lower in the opposite direction (amplitude changes by 3.36 mK).

6.3 Expansion of the Universe

Slipher in 1910 noted that galaxies are all red-shifted so moving away from us.

Hubble discovered that v= $H_0 D$ (law is valid for $D \ll R_H$ or v $\ll c$—it takes a more complicated form for higher v).

6.3.1 Age of universe

If we assume v of any galaxy at distance D is independent of time, then the age of the Universe, τ, is given by $\tau = D/\mathrm{v} = \frac{1}{H_0}$. $\dot{\tau}$ is called the Hubble time.

$$\tau = \frac{1}{100h\ \mathrm{kms}^{-1}\mathrm{Mpc}^{-1}}$$

$$= \frac{3.08 \times 10^{24}\ \mathrm{cm}}{100h\ 10^{5}\mathrm{cm\ s}^{-1}} = 1 \times 10^{10}\ h^{-1}\mathrm{yr} \qquad (6.4)$$

$$\sim 14 \text{ billion years .}$$

6.4 Newtonian Dynamics

Complete description needs General Relativity but for

$$\frac{\mathrm{v}}{\mathrm{c}} << 1 \text{ or } D << \frac{c}{H_0} = 3000\ h^{-1}\ \mathrm{Mpc}$$

and

$$\frac{GM}{Rc^2} << 1 .$$

Newtonian theory is adequate.

Consider an expanding sphere of radius R with a speed $\dot{R} = H_0 R$. There is no gravitational force on the outside. However the surface of the sphere feels a deceleration due to the gravitational attraction of the enclosed mass

$$\ddot{R} = -\frac{GM(R)}{R^2} = \frac{-4\pi}{3} G\rho(t)\, R \qquad (6.5)$$

when ρ is the mass density.

Write $R(t) = R_o a(t)$ where R_o is radius at the present time $t = t_o$ and $a(t_o)=1$. Then

$$\ddot{a}(t) = -\frac{4\pi}{3} G\rho(t)\, a(t) \; . \qquad (6.6)$$

To solve for $a(t)$, we need to know $\rho(t)$.

In a matter dominated Universe, total mass in the sphere is constant:

$$M(R) = \frac{4}{3} \pi R_o^{\,3}\, a^3 \rho \qquad (6.7)$$

so

$$\rho = \rho_0 / a^3 \; . \qquad (6.8)$$

Eqn. (6.6) becomes

$$\ddot{a} = -\frac{4\pi}{3} G\rho_0 \frac{1}{a^2} \; . \qquad (6.9)$$

Multiply by $\dot{a}$

$$\dot{a}\,\ddot{a} = -\frac{4\pi}{3} G\rho_o \frac{\dot{a}}{a^2} \; . \qquad (6.10)$$

Integrate to get

$$\dot{a}^2 = \frac{8\pi}{3} G\rho_0 \frac{1}{a} + \text{constant}$$

$$= \frac{8\pi}{3} G\rho_0 \frac{1}{a} - kc^2 \qquad (6.11)$$

or

$$\dot{a}^2 = \frac{8\pi}{3} G\rho a^2 - kc^2 \tag{6.12}$$

where we write the constant as $-kc^2$

$-k$ has dimension (length)$^{-2}$.

This is Lemaitre equation (or Friedman or Einstein). Depending on whether $k > 0$, $k = 0$ or $k < 0$ we find three kinds of behavior.

$\boldsymbol{k > 0}$ When $kc^2 = \frac{8\pi}{3} \frac{G\rho_o}{a(t)}$ or

$$\text{when} \quad a = \frac{8\pi G\rho_o}{3kc^2}, \text{ then } \dot{a} = 0 . \tag{6.13}$$

Thus expansion stops when $a = \frac{8\pi G\rho_o}{3kc^2}$ and the Universe reaches a maximum size and collapses—a closed Universe.

$\boldsymbol{k = 0}$

$$\dot{a}^2 = \frac{8\pi}{3} \frac{G\rho_o}{a} \tag{6.14}$$

As $a \to \infty$, $\dot{a} \to 0$

Universe expands with a velocity that tends asymptotically to zero—a flat Universe.

$\boldsymbol{k < 0}$ $\dot{a} \to$ a finite value $(-kc^2)^{1/2}$, Universe expands faster

As $a \to \infty$, open Universe.

6.4.1 Critical Density

To distinguish an open or closed Universe, we compare the density with a critical density.

Define Hubble constant $H(t)$ at time t as

$$H(t) = \frac{\mathrm{v}}{D} = \frac{\dot{R}}{R} = \frac{R_o \dot{a}}{R_o a} = \left(\frac{\dot{a}}{a}\right)_t \tag{6.15}$$

Then

$$H(t_0) = H_0 = \left(\frac{\dot{a}}{a}\right)_{t=t_o} = \left(\frac{\dot{a}(t_o)}{a_o}\right) = \dot{a}(t_o) \,. \tag{6.16}$$

From Lemaitre eqn. (6-12)

we get

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3} G\rho - \frac{kc^2}{a^2} \,. \tag{6.17}$$

Define the critical density at time t

$$\rho_c = \frac{3H^2}{8\pi G} \,, \tag{6.18}$$

At $t = t_o$, we write for the density today

$$\rho_c(t_0) = \frac{3H_0{}^2}{8\pi G} = 1.9 \times 10^{-29} \, h^2 g \,\mathrm{cm}^{-3} \,. \tag{6.19}$$

Then write (6-17) in the form (multiply by $3/8\pi G$)

$$\rho = \rho_c + \frac{3kc^2}{8\pi G a^2} \,. \tag{6.20}$$

Introduce the density parameter

$$\Omega(t) = \rho(t)\,/\rho_c \ . \tag{6.21}$$

Then

$$\Omega = 1 + \frac{3kc^2}{8\pi G\rho_c a^2} \tag{6.22}$$

and $\Omega < 1$, $= 1$, > 1 for $k < 0$, $= 0$, > 0 . Ω is a function of time through ρ_c , unless $k = 0$.

The deceleration parameter q is the dimensionless parameter

$$q = \frac{-a\,\ddot{a}}{\dot{a}^2} \ . \tag{6.23}$$

It is a measurement of how fast the Universe is decelerating.

Present value is q_o.

From (6.6), one gets

$$\therefore\ q = -\frac{a\ddot{a}}{\dot{a}^2} = \frac{4\pi}{3}\,G\rho\,\frac{a^2}{\dot{a}^2} \tag{6.24}$$

But from (6.15), $H = \dot{a}/a$. Using (6.18),

$$q = \frac{4\pi G}{3H^2}\,\rho = \frac{1}{2}\frac{\rho}{\rho_c} = \frac{1}{2}\,\Omega \ . \tag{6.25}$$

So for a matter-dominated Universe

$q = \frac{1}{2}$ for a flat Universe

$< \frac{1}{2}$ open

$> \frac{1}{2}$ closed

***k* < 0**	***k* = 0**	***k* > 0**
unbound solution	marginally bound	bound solution
open universe	flat universe	closed universe
$\rho < \rho_c$	$\rho = \rho_c$	$\rho > \rho_c$
$\Omega(t) < 1$	$\Omega(t) = 1$	$\Omega(t) > 1$
$q < 1/2$	$q = 1/2$	$q > 1/2$
infinite volume	infinite volume	finite volume

6.5 Flatness Problem

$$\text{Critical density} = \frac{3H_0^{\;2}}{8\pi G} = 1.9 \times 10^{-29}\ \text{g cm}^{-3}\ .$$

Microwave background has density

$$\rho_k = \frac{\varepsilon_k}{c^2} = \frac{4.23 \times 10^{-13}\text{ergs cm}^{-3}}{c^2} = 4.7 \times 10^{-34}\text{g cm}^{-3} \quad , \quad \Omega_k \sim 10^{-5}$$

Visible stars contribute $\Omega_o = 0.02$

and dark matter $\Omega_o \sim 0.2$

Now from (6.20) $\rho = \rho_c + \dfrac{3kc^2}{8\pi G a^2}$,

$$1 = \frac{\rho_c}{\rho} + \frac{3kc^2}{8\pi G a^2 \rho} = \frac{1}{\Omega} + \frac{3kc^2 a}{8\pi G \rho_o}$$

$$\therefore 1 - \frac{1}{\Omega} = \frac{3kc^2}{8\pi G \rho_o} a \;\; \alpha \;\; a \tag{6.26}$$

In the past $a << a_o$ so over most of time Ω was very close to 1.

Why not now?

6.6 Einstein-deSitter Universe Ω = 1

Consider the variation of ρ.

First law of thermodynamics (adiabatic):

$$dE = -P dV$$

Then

$$\frac{dE}{dt} + P\frac{dV}{dt} = 0\ . \tag{6.27}$$

In a sphere of radius $R(t) = R_o\, a(t)$

$$E = Mc^2 = \frac{4\pi}{3} R_o^3 a^3 \rho c^2 \quad , \quad V = \frac{4}{3}\pi\ R_o^3 a^3\ . \tag{6.28}$$

Thus

$$\frac{d}{dt}\left(\rho c^2 a^3\right) + P\frac{d}{dt}\left(a^3\right) = 0. \tag{6.29}$$

6.6.1 Non-relativistic matter

$P \sim \rho \mathrm{v}^2$ where v << c . Ignore P. Then $\frac{d}{dt}\left(\rho a^3\right) = 0$

$$\rho a^3 = \rho_o a_o^3 \text{ (conservation of particles).} \tag{6.30}$$

6.6.2 Relativistic matter (radiation)

$$P = \frac{1}{3}u = \frac{1}{3}\rho c^2 \tag{6.31}$$

$$\frac{d}{dt}\left(\rho a^3\right) + \frac{1}{3}\rho \frac{d}{dt}\left(a^3\right) = 0 \tag{6.32}$$

which is equivalent to

$$\frac{d}{dt}\left(\rho a^4\right) = 0. \tag{6.33}$$

Proof

$$\frac{d}{dt}\left(\rho a^4\right) = \frac{d}{dt}\left(\rho a^3 \times a\right)$$

$$= a\frac{d}{dt}\left(\rho a^3\right) + \rho a^3 \frac{da}{dt}$$

$$= a\frac{d}{dt}\left(\rho a^3\right) + \rho a\left(a^2 \frac{d}{dt}\right)$$

$$= a\left\{\frac{d}{dt}\left(\rho a^3\right) + \frac{\rho}{3}\frac{d}{dt} a^3\right\} = 0 . \tag{6.34}$$

So

$$\rho a^4 = \text{constant} = \rho_o a_o^4 . \tag{6.35}$$

As Universe expands, number of photons is conserved. Density decreases as a^{-4} and photon energy $h\nu = hc/\lambda$ decreases as a^{-1} through the red shift.

6.6.3 Matter-dominated Einstein-deSitter Universe

Time-dependence of flat Universe

k=0 $\rho a^3 = \rho_o a_o^{\ 3}$

Eqn. 6.14 is

$$\dot{a}^2 = \frac{8\pi G\rho a^2}{3} = \frac{8\pi G\rho_o a_o^{\ 3}}{3}\ \frac{1}{a}$$

$$a^{1/2}\frac{da}{dt} = \left(\frac{8\pi G\rho_o a_o^{\ 3}}{3}\right)^{1/2} \qquad (6.36)$$

$$\frac{2}{3}a^{3/2} = \left(\frac{8\pi G\rho_o a_o^{\ 3}}{3}\right)^{1/2} t\,.$$

Thus

$$\frac{a}{a_o} = \left(6\pi G\rho_o t^2\right)^{1/3} \sim t^{2/3}\,. \qquad (6.37)$$

Using (6.37) in $\rho a^3 = \rho_o a_o^3$,

$$\rho = \frac{1}{6\pi G t^2} \sim t^{-2}\,. \qquad (6.38)$$

With $H = \dot{a}/a$, $t = \frac{2}{3}\ \frac{1}{H}$

Current age is $t_o = \frac{2}{3}\ \frac{1}{H_0}$

$= 6.5 \times 10^9\ h^{-1}$ years.

Note $t = (6\pi G\rho)^{-1/2}$.

6.6.4 Radiation dominated Einstein-de Sitter Universe

Time dependence of Flat Universe

Eqn. 6.35 is

$$\rho a^4 = \rho_o a_o{}^4$$

$$\dot{a}^2 = \frac{8\pi G}{3}\ \rho a^2 = \frac{8\pi G \rho_o a_o{}^4}{3}\ \frac{1}{a^2}$$

$$a\frac{da}{dt} = \left(\frac{8\pi G \rho_o a_o{}^4}{3}\right)^{1/2} \qquad (6.39)$$

$$\frac{1}{2}a^2 = \left(\frac{8\pi G \rho_o a_o{}^4}{3}\right)^{1/2} t$$

$$\therefore\ \frac{a}{a_o} = \left(\frac{32\pi G \rho_o t^2}{3}\right)^{1/4} \sim t^{1/2} \qquad (6.40)$$

$$\rho = \frac{3}{32\pi G t^2} \sim t^{-2}. \qquad (6.41)$$

Then

$$H = \frac{1}{2t} \,, \tag{6.42}$$

$$t = \frac{1}{2H} = \left(\frac{32\pi G\rho}{3}\right)^{-2} . \tag{6.43}$$

6.6.5 Red Shift

Galaxy is moving away. Frequency shift $d\nu$ is given by

$$\frac{d\nu}{\nu} = -\frac{\text{v}}{c} = -\frac{HD}{c} = -\frac{\dot{a}}{a}\frac{cdt}{c} = -\frac{da}{a} . \tag{6.44}$$

So $ad\nu + \nu da = 0$

$d(\nu a) = 0$, νa = constant

So in general

$$\frac{\nu_{obs}}{\nu_{em}} = \frac{\lambda_{em}}{\lambda_{obs}} = \frac{a_{em}}{a_{obs}} < 1 \tag{6.45}$$

red shift z:

$$1 + z = \frac{\lambda_{obs}}{\lambda_{em}} = \frac{R_o}{R} \tag{6.46}$$

$$= \frac{\text{current scale}}{\text{scale when photon was emitted}} .$$

Quasars are seen to $z = 5$ when Universe was 1/6 its present radius and the density was $6^3 \sim 200$ times greater.

For matter-dominated Universe (from 6-37)

$$\frac{t}{t_o} = \left(\frac{a}{a_o}\right)^{3/2} = \frac{1}{(1 + z)^{3/2}} \quad . \tag{6.47}$$

6.7 Cosmological constant

To avoid an expanding Universe, Einstein added a term to his equation of general relativity. It was a repulsive term intended to counteract gravity. Equation (6.5) for $\ddot{R}$ on p. 6.4 is modified to

$$\ddot{R} = -\frac{4\pi G}{3}\rho R + \frac{\Lambda R}{3} \tag{6.48}$$

Λ is the cosmological constant.

If $\Lambda > 0$, additional force is repulsive and if $\Lambda < 0$, it is attractive. It increases with R, whereas the gravity term decreases as R^{-2}.

Then (6.17) becomes

$$\dot{a}^2 = \frac{8}{3}\pi G\rho a^2 + \frac{\Lambda}{3}a^2 - kc^2 \tag{6.49}$$

and (6.25)

$$\Lambda = 4\pi G\rho - 3H^2 q \, . \tag{6.50}$$

Einstein chose $\Lambda = 4\pi G\rho$ to make $H = 0$ for a static universe.

6.8 History of Early Universe—Recombination, Matter/Radiation Decoupling

Initially $T > 10^{10}$K — only fundamental particles and photons — gluon plasma containing quarks, leptons, neutrinos.

After 1 sec, $T = 10^{10}$ K, quarks combined to form protons, protons emitted positrons to form neutrons — $\overline{\nu} + p \rightarrow e^{+} + n$.

At t = 200 seconds, nucleosynthesis occurred

$$p + n \rightarrow d + \gamma$$

$$d + d \rightarrow {}^{3}\mathrm{He} + n \rightleftarrows {}^{3}\mathrm{H} + p$$

$${}^{3}\mathrm{He} + d \rightarrow {}^{4}\mathrm{He} + p$$

$$d + d \rightarrow {}^{4}\mathrm{He} + \gamma . \tag{6.51}$$

The fractional amount of d depended on the baryon density, because of the $d + d$ destruction reactions.

Deuterium was made only in the primordial era. It has been destroyed subsequently by nuclear burning in stars (astration) so the value today is less than the initial value. Fig. 6.1 shows the dependence of the derived baryon density on the D/H ratio. The actual value corrected for astration is about 1×10^{-5}, giving a value for Ω (baryonic) of Ω = 0.1 for h = 0.7.

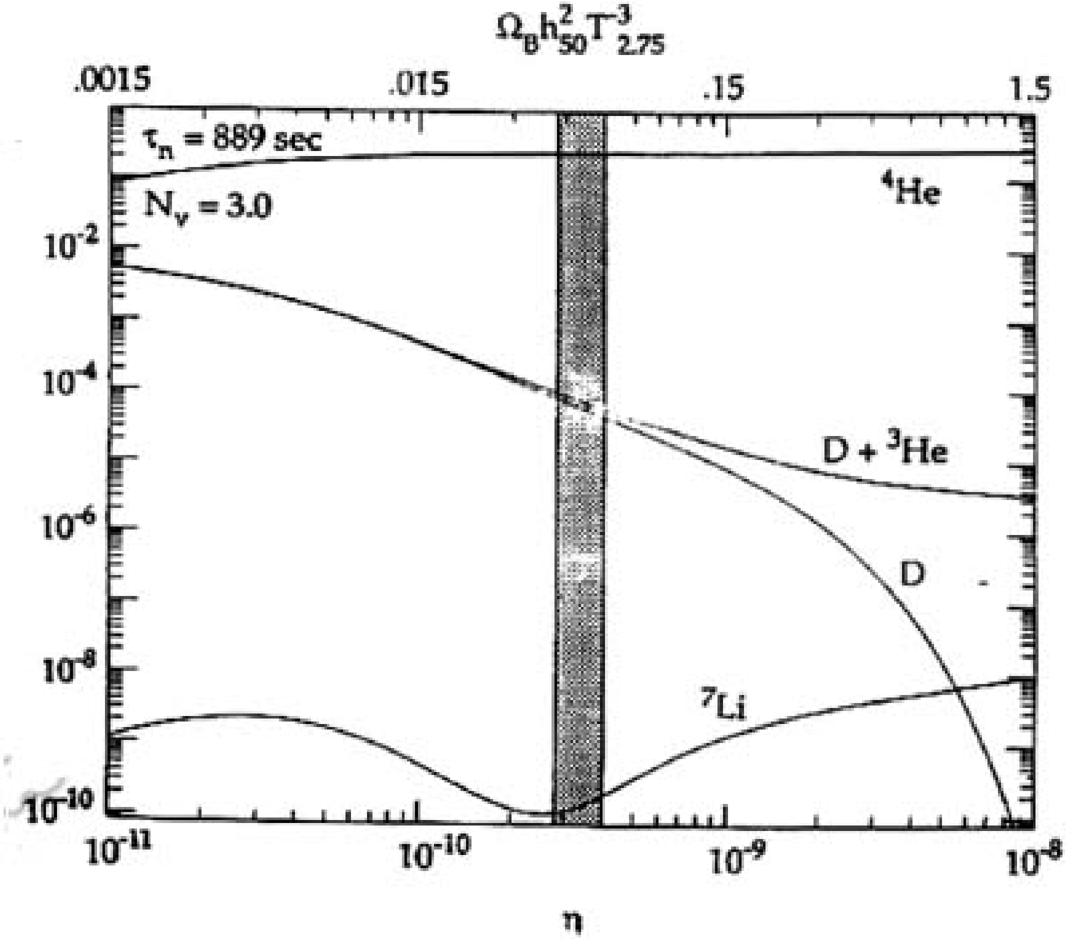

FIG. 12.—Predicted abundances (by number) of D, D + ^{3}He, and ^{7}Li, and the ^{4}He mass fraction as a function of η for $N_\nu = 3$ and $\tau_n = 889$ s for $0.1 \leq \eta_{10} \leq 100$. The vertical band delimits the range of η consistent with the observations. η is the baryon/photon ratio.

Fig. 6.1

^{7}Li was created also by

$$^{4}He + {}^{3}H \rightarrow {}^{7}Li + \gamma$$

$$^{4}He + {}^{3}He \rightarrow {}^{7}Be + \gamma$$

$$^{7}Be \rightarrow {}^{7}Li + e^{+} + \nu \quad . \tag{6.52}$$

But there nucleosynthesis stopped as the Universe grew too cold for reactions to overcome the Coulomb barrier. The Universe then consisted of H^{+} , D^{+} , $^{4}He^{2+}$, $^{7}Li^{3+}$, photons and neutrinos and electrons in a fully ionized electron plasma. Recombination (radiative) such as

$$H^{+} + e \rightarrow H + \nu \tag{6.53}$$

was immediately followed by photoionization by the cosmic photons

$$H + \nu \rightarrow H^{+} + e \tag{6.54}$$

The electrons and photons were coupled by Thompson scattering and shared a common temperature. The Universe continued to cool for about 10^{5} years until at z of ~ 1500, at a temperature of about 4000K, it ran out of photons with enough energy to cause ionization and the recombination era ensued in which the Universe went from fully ionized to neutral except for a few relict electrons. Thermal contact between matter and radiation was lost and matter and radiation evolved independently.

The period between recombination and the formation of galaxies is called the dark ages.