18. Atmospheric scattering details

See *Chandrasekhar* for copious details and also *Goody & Yung* Chapters 7 (Mie scattering) and 8.

Legendre polynomials are often convenient in scattering problems to expand the phase function Φ. The preferred (my preferred) definition for Legendre polynomials is:

$$(2l+1)\int d\Omega[P_l(\cos\theta)]^2 = 4\pi$$

First several:

$$P_0 = 1$$

$$P_1 = \cos\theta$$

$$P_2 = 1/2(3\cos^2\theta - 1)$$

$$P_3 = 1/2(5\cos^3\theta - 3\cos\theta)$$

They form an *orthonormal* basis set. In order to generate them:

$$(l+1)P_{l+1}(\cos\theta) = (2l+1)\cos\theta P_l - lP_{l-1}.$$

Phase function expansion is given in general as:

$$\Phi = \sum_{l=0}^{\infty}(2l+1)m_l P_l(\cos\theta).$$

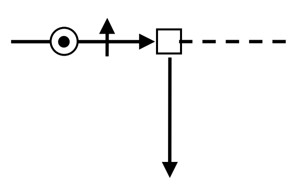
Expansion of the phase function is important in radiative transfer modeling. The required number of expansion terms is limited by the number of terms in the radiative transfer expansion itself (about which more later). For Rayleigh scattering,

$m_0 = 1$
$m_1 = 0$
$m_2 = 1/10$
$m_{>2} = 0$

$$\Phi = \frac{3}{4}(1+\cos^2\theta) = 1 + \frac{1}{10}\times 5 \times \frac{1}{2}(3\cos^2\theta - 1).$$

A Rayleigh Φ with *depolarization* (because of the *Raman* component, as before) is:

$\Phi = \frac{3}{4}\left(\frac{2}{2+\delta}\right)((1+\delta)+(1-\delta)\cos^2\theta)$, where δ is the *depolarization factor* (= 0.0295 for air at 400 nm wavelength). $\delta \equiv \frac{I_H}{I_V} @ \theta = 90^\circ$, for unpolarized input (check that $\delta = 0$ for pure Rayleigh scattering!) for this phase function,



$m_0 = 1$
$m_1 = 0$

$$m_2 = \frac{1}{5}\left(\frac{1-\delta}{2+\delta}\right)$$

$$m_{>2} = 0$$

Mie scattering

Aerosols and clouds, especially. Horribly complicated general solutions, lots of oscillations in phase functions, which average out over size distributions.

Details in *Goody and Yung*, Chapter 7, and in notes from J. Wang.

Note the distinction between absorbing and non-absorbing aerosols: Complex index of refraction, $\hat{m} = \hat{n} - i\hat{n}'$. *(NB black carbon vs. sulfates, clouds)*

Transmission $E = E_0 \exp\left[-2\pi i\left(\frac{d}{\lambda} + \nu t\right)\right]$. Since $\lambda = \lambda_0 / \hat{m}$, $\hat{n}'$ leads to extinction.

A typical Mie setup for computation (W. Wiscombe in Disort test code) has 82 Legendre terms in a typical *haze* and 299 terms in *cloud*.

Mie scattering is strongly *forward-peaked* (tea kettle example), sometimes with a secondary backward, structured peak (a *glory*).

The Henyey-Greenstein phase function is a common practical Mie phase approximation with nice analytic properties:

$\Phi_{HG}(\cos\theta, g) = \dfrac{1-g^2}{\left(1+g^2-2g\cos\theta\right)^{3/2}}$, where g is the *asymmetry parameter*. $g \sim 0.6$ is typical for atmospheric aerosol. This Henyey-Greenstein phase function misses the back scattering peak. This can be treated using the double Henyey-Greenstein phase function: $\Phi = b\Phi_{HG}(\cos\theta, g_1) + (1-b)\Phi_{HG}(\cos\theta, g_2)$, where $g_2 < 0$.

Goody and Yung give a typical atmospheric example (maritime haze @ 0.7 μm):
$g_1 = 0.824, g_2 = -0.55, b = 0.9724$.

For Φ_{HG}, $m_0 = 1$, $m_l = \Phi_{HG}, m_o = 1, m_l = (g)^l$

For the double *HG*, $m_0 = 1, m_l = bg_1^l + (1-b)g_2^l$

Finally, note the weak wavelength-dependence of Mie scattering compared to Rayleigh scattering; it is sometimes $\propto \lambda^{-1}$. *(Or some other low power)*

For Mie scattering by clouds and aerosols, the most common distribution of sizes is *log-normal*: $\frac{dN(r)}{dr} = \frac{c}{\sigma r\sqrt{2\pi}} \exp\left[\frac{-(\ln r - \ln \bar{r})^2}{2\sigma^2}\right]$, where σ is the *shape parameter*, the ln of the standard distribution in width.

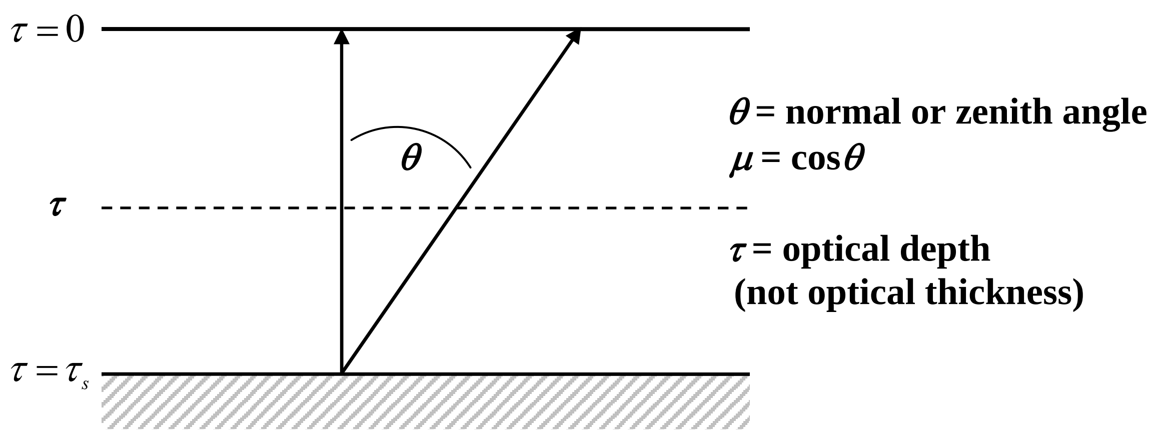
Shettle and Fenn (see references) is a standard source for aerosol information. They describe atmospheric aerosol distributions as either one or the sum of two log-normal distributions.

Details on aerosols to come!

2-Stream, plane-parallel formulation

Recall that $\varepsilon_\sigma = k_\sigma + m_\sigma$ and $m_\sigma / \varepsilon_\sigma = \omega_\sigma$ (single scattering albedo). If $\omega_\sigma = 1$, the scattering is *conservative*.

Setup for plane-parallel atmosphere:



μ is defined as cos (θ) where θ is (often) the zenith angle *ZA*; sometimes the nadir angle ($\pi - ZA$); and even $\mu = |\cos(ZA)|$. We will use $\mu = \cos\theta = \cos(ZA)$. For scattering problems without explicit azimuthal dependence, for an arbitrary scattering function, *G*

$$\int_\Omega G(\Omega)d\Omega = \int_0^{2\pi}\int_0^{\pi} G(\theta,\phi)\sin\theta d\theta d\phi = 2\pi\int_0^{\pi} G(\cos\theta)\sin\theta d\theta = 2\pi\int_{-1}^{1} G(\mu)d\mu,$$

a convenient and useful substitution (which assumes, as is normally the case, that expansion in cos θ is valid.).

Differential scattering event: $I \longrightarrow \square \longrightarrow I - dI$ (just absorption and emission, *k,* for starters)

$dI = -Id\tau + Bd\tau,\quad \frac{-dI}{d\tau} = I - B$ Already in the basic *Schwarzschild* form of a radiative transfer equation (The most basic Schwarzschild form is $\frac{-dI_\nu}{d\tau} = I_\nu - J_\nu$ or, expanding τ, $\frac{-dI_\nu}{k_\nu \rho ds} = I_\nu - J_\nu$). It is thus sometimes preferable to keep a dimension:

$$\frac{-1}{\varepsilon}\frac{dI}{ds} = I - B, \text{ so } d\tau = \varepsilon ds$$

Scattering for layered atmospheres can be formulated with τ as the independent variable (usually) or *z* (or *P*) as the independent variable (sometimes).

τ is easier mathematically but tougher when polychromatic (spectral) problems are addressed because $\tau = \tau(\sigma)$ but $z \neq z(\sigma)$. Also note, however, that when τ changes rapidly (like at a cloud top), using *z* or P may require extra effort (like creating very fine vertical layers).

Add scattering out of beam

scattering source term

$$\begin{aligned} dI &= -I\varepsilon ds + Bkds + mI\Phi(\theta)ds \\ &= -I\varepsilon ds + B(\theta)\varepsilon(1-\omega)ds + I\varepsilon\omega\Phi(\theta)ds. \end{aligned}$$

(Remember $\varepsilon = k + m$.)

Writing $B(\theta)$ formally acknowledges that the blackbody emission has a phase function, even though it is isotropic. Combining source terms (*J* is the standard symbol for the source):

$\varepsilon J \equiv \varepsilon\left[B(\theta)(1-\omega) + I\omega\Phi(\theta)\right]$. Then, $\frac{-1}{\varepsilon}\frac{dI}{ds} = I - J.$

We still need a more complex source (*e.g.*, I_0 might be a pencil beam source or a parallel bundle, like $I_\odot$; $\odot =$ the Sun) and recast with $d\tau = \varepsilon ds$:

$$dI = -Id\tau + B(\Omega)(1-\omega)d\tau + \frac{\omega d\tau}{4\pi}\int d\Omega'\left[I(\Omega')\Phi(\Omega',\Omega)\right].$$

function of relative θ

$$-\frac{dI}{d\tau} = I - B(1-\omega) - \frac{\omega}{4\pi}\int d\Omega' I(\Omega')\Phi(\Omega',\Omega) = I - J.$$

Stratified plane-parallel atmosphere

Goody and Yung 2.3.3; *Chandrasekhar* Chapters I and II

$\boldsymbol{\mu > 0}$ (outward, upward)

$$I^+(\tau,\mu) = I^+(\tau_s,\mu)e^{-(\tau_1-\tau)/\mu} + \int_{\tau_s}^{\tau}\frac{d\tau'}{\mu}J(\tau',\mu)e^{-(\tau'-\tau)/\mu}$$

$\boldsymbol{\mu} < \mathbf{0}$ (inward, downward) + because $\mu < 0$

$$I^-(\tau,\mu) = I^-(0,\mu)e^{+\tau/\mu} - \int_0^{\tau} \frac{d\tau'}{\mu} J(\tau',\mu)e^{+(\tau-\tau')/\mu}$$

Then, for fluxes (*e.g.*, radiative balance), integrate over $d\Omega$ (noting that the *BRDF* of the surface may introduce ϕ-dependence).

For solar radiation and negligible thermal source, calculate I^- and use albedo:
$a_\sigma = \dfrac{F_\sigma^+}{F_\sigma^-}$ at surface (F = flux); then $J_\sigma = \dfrac{a_\sigma}{\pi} F_\sigma^-$ (surface).

Plane parallel, direction from boundary *inward* (or *downward*) See *Chandrasekhar* Chapters I and II for details

$\mu \dfrac{dI}{d\tau} = I - J$ Note *apparent* sign change in radiative transfer equation since μ= < 0 for the downward direction.

$\boldsymbol{\mu}\uparrow \qquad \downarrow\boldsymbol{\tau}$

Two-stream problem (up and down)

Simplest case: isotropic scattering $\Phi = 1$ (*Schuster and Schwartzchild*, from *Chandrasekhar*, leading to later discussions of ***quadrature****)

$dI(\tau,\mu) = \dfrac{I(\tau,\mu)\,d\tau}{\mu} - \dfrac{d\tau}{2\mu}\displaystyle\int_{-1}^{1} \omega I(\tau,\mu')d\mu'$ (the factor of ½ is from integrating over azimuth - 2π and dividing by the isotropic phase function - 4π), or

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu')d\mu'.$$

Then, defining I^+ and I^- from $sign(\mu)$,

$$+\frac{1}{2}\frac{dI^+}{d\tau} = I^+ - \frac{\omega}{2}(I^+ + I^-)$$

$$-\frac{1}{2}\frac{dI^-}{d\tau} = I^- - \frac{\omega}{2}(I^+ + I^-)$$

The extra factors of 1/2 on the left come from averaging both extinction and source terms as $\int_0^{\pi} \sin\theta\cos\theta d\theta / \int_0^{\pi} \sin\theta d\theta = 1/2$ (the *mean obliquity of the rays*).

For conservative scattering (ω = 1, Chandrasekhar, Section 20)

$\frac{dI^+}{d\tau} = \frac{dI^-}{d\tau} = I^+ - I^-$. More realistic: $\Phi \neq 1, \omega \neq 1$.

Gaussian quadrature "seeks to obtain the best numerical estimate of an integral by picking optimal abscissas at which to evaluate the function …"
http://mathworld.wolfram.com/GaussianQuadrature.html

See Chandrasekhar Sections 20-22.