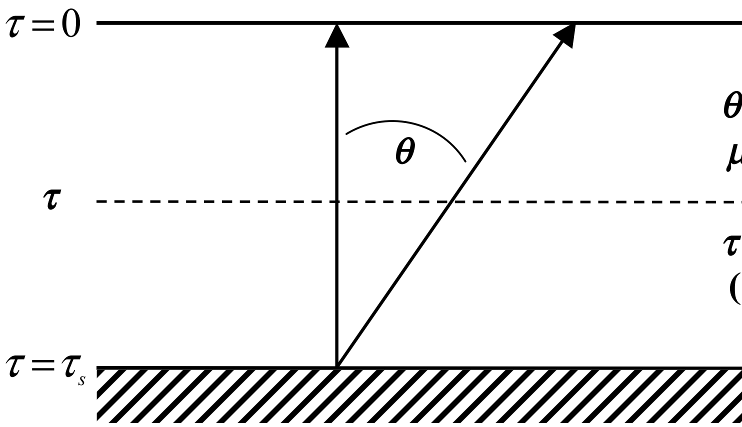
Radiative transfer modeling

2-Stream, plane-parallel formulation

Recall that $\varepsilon_\sigma = k_\sigma + m_\sigma$ and $m_\sigma / \varepsilon_\sigma = \omega_\sigma$ (single scattering albedo). If $\omega_\sigma = 1$, the scattering is *conservative*.

Setup for plane-parallel atmosphere:



θ = normal or zenith angle
$\mu = \cos\theta$

τ = optical depth
(not optical thickness)

μ is defined as cos (θ) where θ is (often) the zenith angle *ZA*; sometimes the nadir angle (π – *ZA*); and even $\mu = |\cos(ZA)|$. We will use $\mu = \cos\theta = \cos(ZA)$. For scattering problems without explicit azimuthal dependence, for an arbitrary scattering function, *G*

$$\int_\Omega G(\Omega) d\Omega = \int_0^{2\pi}\int_0^{\pi} G(\theta,\phi)\sin\theta d\theta d\phi = 2\pi\int_0^{\pi} G(\cos\theta)\sin\theta d\theta = 2\pi\int_{-1}^{1} G(\mu) d\mu,$$

a convenient and useful substitution (which assumes, as is normally the case, that expansion in cos θ is valid.).

Differential scattering event: $I \longrightarrow \square \longrightarrow I - dI$ (just absorption and emission, *k,* for starters)

$dI = -Id\tau + Bd\tau, \quad \frac{-dI}{d\tau} = I - B$ Already in the basic *Schwarzschild* form of a radiative transfer equation (The most basic Schwarzschild form is $\frac{-dI_\nu}{d\tau} = I_\nu - J_\nu$ or, expanding τ, $\frac{-dI_\nu}{k_\nu \rho ds} = I_\nu - J_\nu$). It is thus sometimes preferable to keep a dimension:

$\frac{-1}{\varepsilon}\frac{dI}{ds} = I - B$, so $d\tau = \varepsilon ds$

Scattering for layered atmospheres can be formulated with τ as the independent variable (usually) or z (or P) as the independent variable (sometimes).

τ is easier mathematically but tougher when polychromatic (spectral) problems are addressed because $\tau = \tau(\sigma)$ but $z \neq z(\sigma)$. Also note, however, that when τ changes rapidly (like at a cloud top), using z or P may require extra effort (like creating very fine vertical layers).

Add scattering out of beam

scattering source term

$$\begin{aligned} dI &= -I\varepsilon ds + Bkds + mI\Phi(\theta)\,ds \\ &= -I\varepsilon ds + B(\theta)\varepsilon(1-\omega)ds + I\varepsilon\omega\,\Phi(\theta)\,ds. \end{aligned}$$

(Remember $\varepsilon = k + m$.)

Writing $B(\theta)$ formally acknowledges that the blackbody emission has a phase function, even though it is isotropic. Combining source terms (J is the standard symbol for the source):

$\varepsilon J \equiv \varepsilon\left[B(\theta)(1-\omega) + I\omega\Phi(\theta)\right]$. Then, $\dfrac{-1}{\varepsilon}\dfrac{dI}{ds} = I - J$.

We still need a more complex source (*e.g.*, I_0 might be a pencil beam source (LIDAR) or a parallel bundle, like $I_\odot$; $\odot =$ the Sun) and recast with $d\tau = \varepsilon ds$:

$$dI = -Id\tau + B(\Omega)(1-\omega)d\tau + \frac{\omega d\tau}{4\pi}\int d\Omega'\left[I(\Omega')\Phi(\Omega',\Omega)\right].$$

function of relative θ

$$-\frac{dI}{d\tau} = I - B(1-\omega) - \frac{\omega}{4\pi}\int d\Omega' I(\Omega')\Phi(\Omega',\Omega) = I - J.$$

Stratified plane-parallel atmosphere
Goody and Yung 2.3.3; *Chandrasekhar* Chapters I and II

$\boldsymbol{\mu > 0}$ (outward, upward)

$$I^+(\tau,\mu) = I^+(\tau_s,\mu)e^{-(\tau_1-\tau)/\mu} + \int_{\tau_s}^{\tau}\frac{d\tau'}{\mu}J(\tau',\mu)e^{-(\tau'-\tau)/\mu}$$

$\boldsymbol{\mu < 0}$ (inward, downward) + because $\mu < 0$

$$I^-(\tau,\mu) = I^-(0,\mu)e^{+\tau/\mu} - \int_{0}^{\tau}\frac{d\tau'}{\mu}J(\tau',\mu)e^{+(\tau-\tau')/\mu}$$

Then, for fluxes (*e.g.*, radiative balance), integrate over $d\Omega$ (noting that the ***BRDF*** of the surface may introduce ϕ-dependence).

For solar radiation and negligible thermal source, calculate I^- and use albedo:
$a_\sigma = \dfrac{F_\sigma^+}{F_\sigma^-}$ at surface (F = flux); then $J_\sigma = \dfrac{a_\sigma}{\pi} F_\sigma^-$ (surface).

Plane parallel, direction from boundary *inward* (or *downward*) See *Chandrasekhar* Chapters I and II for details

$\mu \dfrac{dI}{d\tau} = I - J$ Note *apparent* sign change in radiative transfer equation since $\mu < 0$ for the downward direction.

$$\mu \uparrow \qquad \downarrow \tau$$

Two-stream problem (up and down)

Simplest case: isotropic scattering $\Phi = 1$ (*Schuster and Schwartzchild*, from *Chandrasekhar*, leading to later discussions of ***quadrature****)

$dI(\tau,\mu) = \dfrac{I(\tau,\mu)\,d\tau}{\mu} - \dfrac{d\tau}{2\mu}\displaystyle\int_{-1}^{1} \omega I(\tau,\mu')\,d\mu'$ (the factor of ½ is from integrating over azimuth - 2π and dividing by the isotropic phase function - 4π), or

$$\mu \frac{dI(\tau,\mu)}{d\tau} = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} I(\tau,\mu')\,d\mu'.$$

Then, defining I^+ and I^- from $sign(\mu)$,

$$+\frac{1}{2}\frac{dI^+}{d\tau} = I^+ - \frac{\omega}{2}(I^+ + I^-)$$
$$-\frac{1}{2}\frac{dI^-}{d\tau} = I^- - \frac{\omega}{2}(I^+ + I^-)$$

The extra factors of 1/2 on the left come from averaging both extinction and source terms as $\int_0^\pi \sin\theta\cos\theta\,d\theta \,/ \int_0^\pi \sin\theta\,d\theta = 1/2$ (the *mean obliquity of the rays*).

For conservative scattering (ω = 1, Chandrasekhar, Section 20)

$\dfrac{dI^+}{d\tau} = \dfrac{dI^-}{d\tau} = I^+ - I^-$. More realistic: $\Phi \neq 1, \omega \neq 1$.

***Gaussian quadrature** "seeks to obtain the best numerical estimate of an integral by picking optimal abscissas at which to evaluate the function …"

http://mathworld.wolfram.com/GaussianQuadrature.html

See Chandrasekhar Sections 20-22.