Sketch of an approach to replace the radiative transfer integrodifferential equations by a system of linear equations (see *Goody and Yung (GY)*, Chapters 2 and 8)

Expansion of azimuth dependence:

In general, scattering problems have azimuthal (ϕ) dependence, even though Φ may not be explicitly azimuthally-dependent, because of geometry:

$$\mu \frac{dI(\tau,\mu,\phi)}{d\tau} = I(\tau,\mu,\phi) - \frac{\omega(\tau)}{4\pi}\int d\Omega' \Phi(\tau,\mu,\phi,\mu',\phi') I(\tau,\mu',\phi') - \Sigma(\tau,\mu,\phi). \quad (\textit{cf. GY}\ 8.1)$$

$\Sigma(\tau,\mu,\phi)$ is a "primary" source of radiation (*e.g.*, thermal; note that Goody and Yung treat the solar source separately).

a. $\Phi(\tau,\mu,\phi,\mu',\phi')$ is expanded in *spherical harmon*ics, $Y_{lm_l}(\theta,\phi)$, derived from the associated Legendre functions, which now include the ϕ-dependence:

$$Y_{lm_l}(\theta,\phi) = N_{lm_l} P_l^{|m_l|}(\mu) e^{im_l\phi}; \quad P_l^{|m_l|}(\mu) = (1-\mu^2)^{1/2|m_l|} \frac{d^{|m_l|}}{d\mu^{|m_l|}} P_l(\mu).$$

$\Phi(\tau,\mu,\phi,\mu',\phi') = \sum_{l=0}^{N} \alpha_l(\tau) Y_{lm_l}(\theta,\phi),$ Where the number of terms in the expansion in l depends on the anisotropy of the phase function and the degree of accuracy required.

b. I and Σ are expanded in Fourier cosine series in the azimuthal variable ϕ, both up to terms $m = 0,\cdots,N$.

Then we have $N+1$ equations in 2 variables, μ and τ (still integrodifferential), instead of 3 (μ,τ,ϕ):

$$\mu \frac{dI^m}{d\tau}(\tau,\mu) = I^m(\tau,\mu) - \gamma_m \int_{-1}^{1} d\mu' \Phi^m(\tau,\mu,\mu') I^m(\tau,\mu') - \Sigma^m(\tau,\mu), \quad m = 0,\cdots N.$$

Sketch of the Discrete Ordinate Method

Expansion gave us a series of $N+1$ integrodifferential equations in 2 variables.

The use of the *discrete ordinate* expansion gets rid of the integro- part to leave a system of linear differential equations.

Each of our azimuthally-independent equations (we are suppressing *m*-dependence for simplicity) is expanded in $\mu\ (=\cos\theta)$, where the most usual choice is to develop a 2-*n* stream representation with angles at the roots of the corresponding Legendre polynomials, $P_{2n}(\mu)$.

E.g., for a 2-stream expansion, $P_2 = \frac{1}{2}(3\cos^2\theta - 1); |\mu| = 0.57735; |\theta| = 54.7^\circ$

$$P_4(\mu) = \frac{3}{8}\left(\frac{35\mu^4}{3} - 10\mu^2 + 1\right)$$

4-stream: $\mu \pm 1 = 0.3400 \quad \theta \pm 1 = 70.12^\mathrm{o}$

$\mu \pm 2 = 0.8611 \quad \theta \pm 2 = 30.55^\mathrm{o}$

This choice is the *Gaussian quadrature* choice. (Quadrature in general means that a definite integral is being replaced by a sum: See *Wikipedia.*) Gaussian quadrature has the **marvelous** property of being *exact* for Φ = a polynomial of degree ≤ 4*n* (that is, for a 2*n* representation!) for integrated fluxes and intensities.

Expansion gives

$$\mu_{\pm i}\frac{dI}{d\tau}(\tau,\mu_{\pm i}) = I(\tau,\mu_{\pm i}) - \frac{\gamma}{2}\sum_{j=1}^{n} a_j \Phi(\tau,\mu_{\pm i},\mu_j) I(\tau,\mu_j) -$$
$$\frac{\gamma}{2}\sum_{j=1}^{n} a_j \Phi(\tau,\mu_{\pm i},\mu_{-j}) I(\tau,\mu_{-j}) - \Sigma(\tau,\mu_{\pm i}), \quad i = 1, n.$$

The expansion coefficients are given by the Gauss quadrature formula:

$$a_j = \frac{1}{P_m'(\mu_j)}\int_{-1}^{1}\frac{P_m(\mu)d\mu}{\mu - \mu_j}, \text{ where } P_m'(\mu_j) = \left(\frac{dP_m}{d\mu}\right)_{\mu=\mu_j}.$$

These are tabulated extensively (see *Chandrasekhar* Chapter II and Table III), although they may now be easily computed as needed. There are other quadrature formulae, but they do not give results accurate to $\Phi \le 4n$.

We have now replaced our integrodifferential equation in 3 variables with a set of linear differential equations which may be solved by standard methods.

Proceed by setting up a layered atmosphere with $\varepsilon, \omega, \Phi$ for each layer (interpolate from layered in *z* or *P* if necessary to layered in $\tau, \tau = \tau(\sigma)$). This adds an extra dimension (# layers) to the problem: complicated boundary value problem. It can also become complicated when τ changes rapidly.

Other complications:

1. Non-homogeneous terms (*e.g.*, beam source);
2. Strongly-peaked Φs may require other choice for discretization (DISORT and LIDORT discuss this)
3. Output at other than stream angles – uses a complicated (but accurate) interpolation formula *or* put in an extra stream in the calculation with zero weight (see DISORT

and LIDORT). The most basic use is for flux and intensity integrals (see *Goody and Yung*, Chapters 2 and 8).

For a single homogeneous layer,

$$I(\tau,\mu_i) = \sum_{j=-n}^{n} L_j g_j(\mu_i) e^{-k_j \tau} \text{ (homogeneous)} + I_p(\tau,\mu_i) \text{ (particular solution):}$$

Solution to 2*n* first-order differential equations with constant coefficients, plus non-homogeneous terms, where the k_j and g_j are the eigenvalues and eigenvectors of the solution to the differential equations based on the discrete ordinate expansion (*cf. GY* 8.30).

The multiple-layer solution is then a complicated boundary-value problem where the intensity for each azimuthal component and stream angle must be continuous across layer interfaces.

DISORT is the standard discrete ordinate development. It is widely-used and generally available (see class website for references).

LIDORT (developed at the CfA by Rob Spurr, since founder of RT Solutions, Inc) adds calculation of the full Jacobian by a full analytical perturbation analysis of intensity field: Yields Jacobians (weighting functions) in one pass (no finite-differencing); pseudo-spherical and quasi-spherical versions available; surface BRDF; vector (polarization) version available (VLIDORT). Availability: http://www.rtslidort.com/.

There are *many* other approaches:
- Doubling and adding method (*e.g.*, DAK)
- Successive orders of scattering
- Monte Carlo methods
- ….

See *Goody and Yung*, Chapter 8 for details.