Problem set I. Solutions

1. *The mean density of the Sun (1.4 $g\,cm^{-3}$) is higher than the density of water. Why do we still consider the Sun to be a gas sphere all throughout its interior? Derive the condition for this statement in the question as a function of temperature and density for a purely hydrogen plasma. [HINT: Think of the type of interaction between the charged particles inside a star that would restrict their freedom of motion.]*

In liquid water, individual molecules almost touch each other. At the high pressures inside the Sun, hydrogen (and helium) are fully ionized. As a result, the fraction of the the volume taken up by particles (from atoms to nuclei/protons) decreases from about unity to $(r_n/r_a)^3 \approx 10^{-14}$, where $r_n \approx 10^{-13}$ is the nuclear scale and $r_a \approx 0.5 \cdot 10^{-8}$ is the atomic scale. This is one reason why fully ionized matter can remain gaseous up to very high densities.
However, in characterizing the state of stellar matter one has to consider the temperature, as well as the density. For the nuclei to have freedom of motion, it is necessary that their kinetic energy kT be substantially higher than the Coulomb interaction energy. The latter is roughly $e^2/\overline{r}$ where $\overline{r}$ is the average separation between particles. Then the condition $kT \gg e^2/\overline{r}$ can be expressed as

$$T \gg \frac{e^2}{k\,m_p^{1/3}}\rho^{1/3}$$

by making use of relations for a pure hydrogen plasma: $n\overline{r}^3 \sim 1$ and $\rho \sim m_p n$, where n is the concentration of the nuclei/protons. The above condition is re-written as

$$7\,T_6\rho^{-1/3} \gg 1$$

(with $T_6 = T/10^6$). For the center of the Sun $T_6 = 15$ and $\rho = 150\ \mathrm{g\,cm^{-3}}$, so $7\,T_6\,\rho^{-1/3} \approx 20 \gg 1$. So even in its central regions the solar matter constitutes a gas.

2. *Here is a "science fiction like" problem: When interstellar flight became reality, people discovered an extra-solar planet made of unusual stuff. The study of samples from the planet showed that the material compressed under pressure. The pressure, P, and the density, ρ, were found to be related as $P = K\,\rho^2$, where K is a constant coefficient derived in the lab.*
The properties of this material indicated that it is really priceless. Therefore people decided to mine it in cosmic proportions from that distant planet. As soon as they began mining, they dicovered something incredible - despite the huge amounts of material they removed from the planet, its size remained exactly the same.
Show that such a "miracle" is possible, and find out the radius of that planet.

First we can do a brief dimensional analysis of the quantities involved in the problem. The parameters are: the "planet" mass M, its radius R, the constant K, and the gravitational constant G. We denote the dimensionality of a quantity Q by $[Q]$. We then have $[P] = [K\,\rho^2] = [K\,(M/R^3)^2]$. On the other hand, the Newtonian force of gravity, GM/R^2, applied to the surface area of a sphere with radius R also has the dimensionality of pressure: $[P] = [G\,M^2/R^4]$. The ratio of the two expressions will be a dimensionless quantity, α^2:

$$\alpha^2 = \left(\frac{G\,M^2}{R^4}\right)\left(\frac{K\,M^2}{R^6}\right)^{-1}$$

Then

$$R = \alpha \left(\frac{K}{G}\right)^{1/2}.$$

We can expect α to be of order unity.

The above result is quite remarkable - the radius of a self-gravitating equilibrium object with an EOS of $P = K\,\rho^2$ is uniquely determined by K, and its mass is dropped out. Then, for such matter, one could cram *any* amount of mass into the given volume. Of course there will be an upper limit because the escape velocity $V_e = (2\,G\,M/R)^{1/2}$ will grow with $M^{1/2}$ and when it becomes close to c relativistic effects will need to be accounted for.

Let's do a more detailed analysis and derive a value for α. The equilibrium of the self-gravitating spherically symmetric "planet" is expressed as:

$$\frac{dP}{dr} = -\rho \frac{GM_r}{r^2}$$

With $P = K\,\rho^2$ we have

$$\frac{d\rho}{dr} = -\frac{G}{2\,K}\,\frac{M_r}{r^2},$$

where M_r is the mass inside a sphere of radius r, and

$$\frac{dM_r}{dr} = 4\,\pi\,r^2\,\rho,$$

which leads to

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\rho}{dr}\right) = -2\,\pi\frac{G}{K}\rho.$$

If we substitute $y = r\,\rho$, we get an equation of the form

$$\frac{d^2y}{dr^2} + \omega^2\,y = 0$$

where $\omega^2 = 2\,\pi\,G/K$, and the general solution is

$$y = A\,\sin\omega\,r + B\,\cos\omega r$$

where A and B are arbitrary constants. For $r = 0$ the value of $y = r\,\rho$ will be 0 and hence $B = 0$. Therefore

$$\rho = \frac{A}{r}\sin\left(r\sqrt{2\,\pi\,\frac{G}{K}}\right).$$

On the surface where $r = R$, we should have $\rho = 0$, hence

$$R\sqrt{2\,\pi\,\frac{G}{K}} = \pi$$

and

$$R = \sqrt{\frac{\pi}{2}}\sqrt{\frac{K}{G}}$$

which gives us the order-of-unity parameter $\alpha = \sqrt{\pi/2}$.

The bottom line is that we are dealing with an EOS which is precisely such so the self-gravitating matter compresses with the addition of mass to *exactly* compensate for the increase in radius. If the matter were incompressible, the radius of the object would grow as $M^{1/3}$. So, in our case we are dealing with a polytrope of index $n = 1$, where for any polytrope in general

$$P = K \rho^{1+\frac{1}{n}}$$

The polytrope of incompressible matter has an index $n = 0$, and for any $0 < n < 1$ the addition of mass leads to an increase of the radius. For $n > 1$ the addition of mass will lead to a *decrease* in radius, and that is the case for white dwarfs ($n = 3/2$).

3. *Determine the ratio of the number of photons and neutrinos emitted by the Sun per second. Use the fact that during the synthesis of one alpha particle about 26.7 MeV of energy is released, and neutrinos carry away only about 2% of that energy. Assume that the Sun emits that energy as a blackbody when you estimate the average energy per (blackbody) photon.*

The Sun is a blackbody with $T = T_{\rm eff} = 5800$ K, which allows us to estimate the average energy per photon, as $\overline{E}_\gamma = 2.70\, k\, T$. The latter is derived by dividing the energy density

$$u(T) = \frac{4\,\pi}{c} \int_0^\infty B_\nu(T) d\nu$$

of the blackbody radiation by the number of photons per unit volume

$$n_\gamma(T) = \frac{4\,\pi}{c} \int_0^\infty \frac{B_\nu(T)}{h\,\nu} d\nu$$

Thus we find $\overline{E}_\gamma = C\, k\, T$, where if $x \equiv h\,\nu / k\,T$ we have

$$C = \left(\int_0^\infty \frac{x^3\, dx}{e^x - 1} \right) \left(\int_0^\infty \frac{x^2\, dx}{e^x - 1} \right)^{-1}$$

and $\overline{E}_\gamma = 2.70\, k\, T$.
So this corresponds to $2.70 \times 1.38 \cdot 10^{-16} \times 5800 \approx 2.16 \cdot 10^{-12}$erg $= 1.35$ eV. Therefore we get about $2 \cdot 10^7$ photons per synthesis of a single α particle (0.98×26.7MeV$/1.35$eV). On the other hand, only 2 neutrinos are produced. The ratio is then $\sim 10^7$.