Problem set III. Solutions

1. *(30 pts) The density and temperature in the center of the Sun are 150 g cm*$^{-3}$ *and* $15 \cdot 10^6$ *K, respectively. Which is larger: the number of photons or the number of protons ? (Give an order of magnitude estimate.)*

First we estimate the particle number density per unit volume as:

$$n = \frac{\rho}{\mu \, m_p}$$

where the mean molecular weight μ can be estimated approximately as follows: for a fully ionized hydrogen plasma we have two particles (p + e$^-$) per unit of atomic mass ($\approx$ one hydrogen atom), and since $M_e \ll m_p$, then $\mu \approx 0.5$. When you add helium to the mixture, $\mu \approx 0.6$. Therefore $n \approx 10^{26}$ cm^{-3}.
For the photons, which are well described by a blackbody radiation field at $T = 1.5\,10^7$ K,

$$n_\gamma = \frac{4\pi}{c} \int \frac{B_\nu(T)}{h\nu} \, d\nu$$

Solving an integral of the type $C = \int_0^\infty \frac{x^2 \, dx}{e^x - 1}$ in terms of a Riemann $\zeta(s)$ function, gives the familiar expresion

$$n_\gamma = 20 \, T^3$$

Hence in the center of the Sun, $n_\gamma \approx 7 \cdot 10^{22}$ and the photons are far fewer than the particles anywhere in the interior.

2. *(70 pts) The solar atmosphere is gray to a first approximation, i.e. the extinction of visible light by the atmosphere is independent of its wavelength. However, the observed limb darkening of the solar disk increases at shorter wavelengths. Explain why. [HINT: Use blackbody radiation; it is enough to use a power law approximation for the frequency-dependent Planck function.]*

Near the solar limb we see visible radiation which has passed through a relatively long, yet slanted, path in the atmosphere. Therefore it emerges from higher levels in the solar atmosphere with temperature T_0. The radiation from the center of the disk comes from deeper in the atmosphere, where the temperature is T_1. We can safely assume that near the solar photsphere the radial temperature gradient is negative, hence $T_1 > T_0$. Adopting blackbody radiation for the averaged visible continuum intensity, $B_\nu(T_1) > B_\nu(T_0)$, and the ratio $B_\nu(T_0)/B_\nu(T_1)$ gives us the amount of limb darkening.
In a gray atmosphere, extinction is independent of wavelength, therefore the depth of the emergent radiation will also remain the same. While this means that T_0 and T_1 above are also independent of wavelength (or frequency ν), it does not mean that the amount of limb darkening $B_\nu(T_0)/B_\nu(T_1)$ is going to be independent of ν.
To see that, we may use the fact that T_1 is not very different from T_0 so we can approximate $B_\nu(T)$ in the vicinity of $T \equiv T_0$ as:

$$B_\nu(T) \approx B_\nu(T_0) \left(\frac{T}{T_0} \right)^\alpha$$

where

$$\alpha = \frac{h\nu}{kT} (1 - e^{-h\nu/kT})^{-1}$$

Such a power law approximation is easily derivable for a function $f(x)$ which is differentiable at $x = x_0$, if you study $\ln f(x)$ as a function of $\ln x$. The standard linearization in the vicinity of $x = x_0$ is

$$\ln f(x) = lnf(x_0) + \left.\frac{d\ln f(x)}{d\ln x}\right|_{x=x_0} (\ln x - \ln x_0) + \cdots$$

which leads us to

$$f(x) \approx f(x_0) \left(\frac{x}{x_0}\right)^{\alpha}$$

where

$$\alpha = \alpha(x_0) = \left.\frac{d\ln f(x)}{d\ln x}\right|_{x=x_0}$$

Going back to the solar limb darkening,

$$\frac{B_\nu(T_1)}{B_\nu(T_0)} \approx \left(\frac{T_1}{T_0}\right)^{\alpha}$$

We can make at least three points. First the amount of limb darkening (and center-to-limb variations in general) is a function of the temperature gradient in the atmosphere – a steeper gradient leads to stronger limb darkening. Second, for a fixed T gradient, the amount of limb darkening could be different in different spectral regions because of α. Finally, in the spectral region of interest to the problem – the visible –, $\alpha \approx h\nu/kT \gg 1$, and the amount of limb darkening will increase towards shorter wavelengths. Obviously in the mid- and far-infrared ($h\nu/kT \ll 1$) the amount of limb darkening is simply due to T_1/T_0, i.e. small, with no wavelength dependence. These conclusions are applicable to many stars of different mass and temperature.